Generalized equations and their solutions in the (S,0)⊕(0,S) representations of the Lorentz group

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Abstract In this paper I present three explicit examples of generalizations in relativistic quantum mechanics. First of all, I discuss the generalized spin-1/2 equations for neutrinos. They have been obtained by means of the Gersten-Sakurai method for derivations of arbitrary-spin relativistic equations. Possible physical consequences are discussed. Next, it is easy to check that both Dirac algebraic equation $Det(\hat{p}-m)=0$ and $Det(\hat{p}+m)=0$ for $u-$ and $v-$ 4-spinors have solutions with $p_0=\pm E_p=\pm\sqrt{\mathbf{p}^2+m^2}$. The same is true for higher-spin equations. Meanwhile, every book considers the equality $p_0=E_p$ for both $u-$ and $v-$ spinors of the $(1/2,0)\oplus(0,1/2))$ representation only, thus applying the Dirac-Feynman-Stueckelberg procedure for eliminating negative-energy solutions. The recent Ziino works (and, independently, the articles of several others) show that the Fock space can be doubled. We re-consider this possibility on the quantum field level for both $S=1/2$ and higher spin particles. The third example is: we postulate the non-commutativity of 4-momenta, and we derive the mass splitting in the Dirac equation. The applications are discussed.

1 Generalized neutrino equations

A. Gersten [1] proposed a method for derivations of massless equations of arbitrary-spin particles. In fact, his method is related to the van der Waerden-Sakurai [2] procedure for the derivation of the massive Dirac equation. I commented on the derivation of the Maxwell equations in [3]. Then, I showed that the method is rather ambigious because instead of free-space Maxwell equations, one can obtain *generalized* $S=1$ equations, which connect the antisymmetric tensor field with additional scalar

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fields. The problem of physical significance of additional scalar chi-fields should be solved of course by experiment.

In the present paper I apply the van der Waerden-Sakurai-Gersten procedure to spin-1/2 fields. As a result one obtains equations which *generalize* the well-known Weyl equations. However, these equations are known for a long time [4]. Raspini [5, 6] analyzed them again in detail. I add some comments on physical contents of the generalized spin-1/2 equations. The generalized equation can be written in the covariant form.

$$\left[i\gamma^\mu \partial_\mu - \frac{m_2^2 c}{m_1 \hbar}\frac{(1-\gamma^5)}{2} - \frac{m_1 c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi = 0\,. \tag{1}$$

The standard representation of γ^μ matrices has been used here. If $m_1 = m_2$ we can recover the standard Dirac equation. As noted in [4b] this procedure can be viewed as the simple change of the representation of γ^μ matrices. However, this is valid unless $m_2 \neq 0$. Otherwise, entries in the transformation matrix become singular. Furthermore, one can either repeat a similar procedure (the modified Sakurai procedure) starting from the *massless* equation (4) of [1a] or put $m_2 = 0$ in eq. (1). The *massless equation* is

$$\left[i\gamma^\mu \partial_\mu - \frac{m_1 c}{\hbar}\frac{(1+\gamma^5)}{2}\right]\Psi = 0\,. \tag{2}$$

It is necesary to stress that the term '*massless*' is used in the sense that $p_\mu p^\mu = 0$. Then we may have different physical consequences following from (2) comparing with those which follow from the Weyl equation. The mathematical reason of such a possibility of different massless limits is that the corresponding change of representation of γ^μ matrices involves mass parameters m_1 and m_2 themselves. It is interesting to note that we can also repeat this procedure for other definitions, which gives us yet another equation in the massless limit ($m_4 \to 0$):

$$\left[i\gamma^\mu \partial_\mu - \frac{m_3 c}{\hbar}\frac{(1-\gamma^5)}{2}\right]\tilde{\Psi} = 0\,, \tag{3}$$

differing in the sign at the γ_5 term.

The above procedure can be generalized to *any* Lorentz group representations, i.e., to any spins. Is the physical content of the generalized $S = 1/2$ *massless* equations the same as that of the Weyl equation? Our answer is 'no'. The excellent discussion can be found in [4a,b]. First of all, the theory does *not* have chiral invariance. Those authors call the additional parameters as measures of the degree of chirality. Apart from this, Tokuoka introduced the concept of gauge transformations (not to be confuses with phase transformations) for the 4-spinor fields. He also found some strange properties of the anti-commutation relations (see Sec. 3 in [4a] and cf. [8]). And finally, the equation (2) describes *four* states, two of which answer for the positive energy $E = |\mathbf{p}|$, and two others answer for the negative energy $E = -|\mathbf{p}|$.

I just want to add the following to the discussion. The operator of the *chiral-helicity* $\hat{\eta} = (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})$ (in the spinorial representation) used in [4b] does *not* commute, e.g., with the Hamiltonian of the equation (2). Do not confuse with the Dirac Hamiltonian!

$$[\mathscr{H}, \boldsymbol{\alpha}\cdot\hat{\mathbf{p}}]_- = 2\frac{m_1 c}{\hbar}\frac{1-\gamma^5}{2}(\boldsymbol{\gamma}\cdot\hat{\mathbf{p}})\,. \qquad (4)$$

For eigenstates of the *chiral-helicity* the system of corresponding equations can be read ($\eta = \uparrow, \downarrow$)

$$i\gamma^\mu \partial_\mu \Psi_\eta - \frac{m_1 c}{\hbar}\frac{1+\gamma^5}{2}\Psi_{-\eta} = 0\,. \qquad (5)$$

The conjugated eigenstates of the Hamiltonian $|\Psi_\uparrow + \Psi_\downarrow >$ and $|\Psi_\uparrow - \Psi_\downarrow >$ are connected, in fact, by γ^5 transformation $\Psi \to \gamma^5\Psi \sim (\boldsymbol{\alpha}\cdot\hat{\mathbf{p}})\Psi$ (or $m_1 \to -m_1$). However, the γ^5 transformation is related to the PT ($t \to -t$ only) transformation [4b], which, in its turn, can be interpreted as $E \to -E$, if one accepts the Stueckelberg idea about antiparticles. We associate $|\Psi_\uparrow + \Psi_\downarrow >$ with the positive-energy eigenvalue of the Hamiltonian $E = |\mathbf{p}|$ and $|\Psi_\uparrow - \Psi_\downarrow >$, with the negative-energy eigenvalue of the Hamiltonian ($E = -|\mathbf{p}|$). Thus, the free chiral-helicity massless eigenstates may oscillate to one another with the frequency $\omega = E/\hbar$ (as the massive chiral-helicity eigenstates, see [7a] for details). Moreover, a special kind of interaction which is not symmetric with respect to the chiral-helicity states (for instance, if the left chiral-helicity eigenstates interact with the matter only) may induce changes in the oscillation frequency, like in the Wolfenstein (MSW) formalism.

2 Negative energies in the Dirac equation

The general scheme for constructing the field operator has been presented in [9]. During the calculations above we had to represent $1 = \theta(p_0) + \theta(-p_0)$ in order to get positive- and negative-frequency parts. Moreover, during these calculations we did not yet assume which equation this field operator (namely, the $u-$ spinor) does satisfy, with negative- or positive- mass? In general we should transform $u_h(-p)$ to the $v(p)$. The procedure is the following; see [10]. In the Dirac case we should assume the following relation in the field operator:

$$\sum_h v_h(p) b_h^\dagger(p) = \sum_h u_h(-p) a_h(-p)\,. \qquad (6)$$

By direct calculations, we find that

$$-mb_{(\mu)}^\dagger(p) = \sum_\lambda \Lambda_{(\mu)(\lambda)}(p) a_{(\lambda)}(-p)\,. \qquad (7)$$

Hence, $\Lambda_{(\mu)(\lambda)} = -im(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$, and

$$b^{\dagger}_{(\mu)}(p) = i\sum_{\lambda}(\boldsymbol{\sigma}\cdot\mathbf{n})_{(\mu)(\lambda)}a_{(\lambda)}(-p)\,. \tag{8}$$

However, other ways of thinking are possible. Unless the unitary transformations do not change the physical content, we have that the negative-energy spinors $\gamma^5\gamma^0 u^-$ satisfy the accustomed "positive-energy" Dirac equation. We should then expect the same physical content. Their explicite forms $\gamma^5\gamma^0 u^-$ are different from the textbook "positive-energy" Dirac spinors. They are the following

$$\tilde{u}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix}-p^+ + m\\ -p_r\\ p^- - m\\ -p_r\end{pmatrix}, \tag{9}$$

$$\tilde{\tilde{u}}(p) = \frac{N}{\sqrt{2m(-E_p+m)}}\begin{pmatrix}-p_l\\ -p^- + m\\ -p_l\\ p^+ - m\end{pmatrix}. \tag{10}$$

$E_p = \sqrt{\mathbf{p}^2+m^2} > 0$, $p_0 = \pm E_p$, $p^{\pm} = E \pm p_z$, $p_{r,l} = p_x \pm i p_y$. Their normalization is to $(-2N^2)$. Similar formulations have been presented in Refs. [11], and [12]. The group-theoretical basis for such doubling has been given in the papers of Gelfand, Tsetlin and Sokolik [13], who first presented the theory in the 2-dimensional representation of the inversion group in 1956 (later called as "the Bargmann-Wightman-Wigner-type quantum field theory" in 1993). The Markov equations, of course, can be identified with equations for the Majorana-like $\lambda-$ and $\rho-$, which we presented in Ref. [7]. Neither of them can be regarded as the Dirac equation. However, they can be written in the 8-component form as follows:

$$\left[i\Gamma^\mu\partial_\mu - m\right]\Psi_{(+)}(x) = 0\,, \tag{11}$$

$$\left[i\Gamma^\mu\partial_\mu + m\right]\Psi_{(-)}(x) = 0\,. \tag{12}$$

One can also re-write the above equations into two-component forms. Thus, one obtains the Feynman-Gell-Mann [14] equations. As Markov wrote himself, he was expecting "new physics" from these equations. Barut and Ziino [12] proposed yet another model. They considered γ^5 operator as the operator of the charge conjugation. Thus, the charge-conjugated Dirac equation has a different sign in comparison with the ordinary formulation, and the so-defined charge conjugation applies to the whole system, fermion + electromagnetic field, $e \rightarrow -e$ in the covariant derivative. Superpositions of the Ψ_{BZ} and Ψ^c_{BZ} also give us the "doubled Dirac equation", as the equations for $\lambda-$ and $\rho-$ spinors. The concept of the doubling of the Fock space has been developed in the Ziino works (cf. [13, 15]) in the framework of quantum field theory. In their case the self/anti-self charge conjugate states are simultaneously the eigenstates of the chirality. Finally, I would like to mention that in general, in the Weyl basis, the $\gamma-$ matrices are *not* Hermitian, $\gamma^{\mu^\dagger} = \gamma^0\gamma^\mu\gamma^0$. So, $\gamma^{i^\dagger} = -\gamma^i$, $i = 1,2,3$, the pseudo-Hermitian matrix. The energy-momentum operator $i\partial_\mu$ is obviously Hermitian. So, the question is whether the eigenvalues of the Dirac operator $i\gamma^\mu\partial_\mu$ (the mass, in fact) would be always real? The question of the complete system of the eigenvectors of the *non*-Hermitian operator deserve careful consideration [16]. Bogoliubov and Shirkov [9, p.55-56] used the scheme to construct a complete set of solutions of the relativistic equations, fixing the sign of $p_0 = +E_p$.

The main points of this section are: there are "negative-energy solutions" in that is previously considered as "positive-energy solutions" of relativistic wave equations, and vice versa. Their explicit forms have been presented in the case of spin-1/2. Next, relations to previous works have been found. For instance, the doubling of the Fock space and the corresponding solutions of the Dirac equation obtained additional mathematical bases. Similar conclusion can be deduced for higher-spin equations.

3 Non-commutativity in the Dirac equation

The non-commutativity [17, 18] exibits interesting peculiarities in the Dirac case. We analyzed Sakurai-van der Waerden method of derivations of the Dirac (and higher-spins too) equation [19]. We can start from

$$(EI^{(4)} + \boldsymbol{\alpha}\cdot\mathbf{p} + m\beta)(EI^{(4)} - \boldsymbol{\alpha}\cdot\mathbf{p} - m\beta)\Psi_{(4)} = 0\,. \qquad (13)$$

Obviously, the inverse operators of the Dirac operators of the positive- and negative-masses exist in the non-commutative case. We postulate the non-commutativity relations for the components of 4-momenta: $[E, \mathbf{p}^i]_- = \Theta^{0i} = \theta^i$ as usual. Thus, we come to

$$\left\{E^2 - \mathbf{p}^2 - m^2 - (\boldsymbol{\alpha}\cdot\boldsymbol{\theta})\right\}\Psi_{(4)} = 0\,. \qquad (14)$$

However, let us apply the unitary transformation. It is known [7, 20] that one can

$$U_1(\boldsymbol{\sigma}\cdot\mathbf{a})U_1^{-1} = \sigma_3|\mathbf{a}|\,. \qquad (15)$$

The explicit form of the U_1 matrix can be found in [19, 20].

Let us apply the second unitary transformation:

$$\mathscr{U}_2\alpha_3\mathscr{U}_2^\dagger = \begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix}\alpha_3\begin{pmatrix} 1&0&0&0\\ 0&0&0&1\\ 0&0&1&0\\ 0&1&0&0 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 0&1&0&0\\ 0&0&-1&0\\ 0&0&0&-1 \end{pmatrix}. \qquad (16)$$

The final equation is

$$[E^2 - \mathbf{p}^2 - m^2 - \gamma^5_{chiral}|\boldsymbol{\theta}|]\Psi'_{(4)} = 0\,. \qquad (17)$$

In the physical sense this implies the mass splitting for a Dirac particle over the non-commutative space $m_{1,2} = \pm\sqrt{m^2 \pm \theta}$. This procedure may be attractive for explaining the mass creation and mass splitting for fermions.

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