Lie point symmetries of the geodesic equations of the Gödel's metric

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Lie point symmetries of the geodesic equations of the Gödel's metric are found. These form a ten-dimensional Lie algebra. The Lie algebra contains a maximal seven-dimensional solvable sub-algebra. It also contains five dimensional subalgebra of isometries of the metric. The isometries are used to reduce the order of the geodesic system by one. The time-like trajectories of the Gödel's metric are then derived and their graphs in the (r, ϕ) plane are displayed showing some interesting features of the dynamics in this universe.

Finding Lie point symmetries

$$X = \xi(s, x^i)\frac{\partial}{\partial s} + \eta^i(s, x^i)\frac{\partial}{\partial x^i}, \tag{1}$$

for a system of k second order ODEs

$$E^\alpha(s, x^i, \dot{x}^i, \ddot{x}^i) = 0, \quad \alpha, i = 1, ..., k \tag{2}$$

means finding the general solution $\xi(x, y^i)$ and $\eta^i(x, y^i)$ of the determining equations obtained from the symmetry condition [2]

$$\hat{X}(E) = 0, \tag{3}$$

where $\hat{X}$ is the extension of the symmetry operator X written as

$$\hat{X} = \xi(s, x^i)\frac{\partial}{\partial s} + \eta^i(s, x^i)\frac{\partial}{\partial x^i} + (\eta^i)'(s, x^i)\frac{\partial}{\partial \dot{x}^i} + (\eta^i)''(s, x^i)\frac{\partial}{\partial \ddot{x}^i}, \tag{4}$$

and $(\eta^i)'$ and $(\eta^i)''$ are obtained from the extension formula given by

$$(\eta^i)^{(n)} = \frac{d(\eta^i)^{(n-1)}}{ds} - (y^i)^{(n-1)}\frac{d\xi}{ds}. \tag{5}$$

Applying the symmetry condition (3) on each ODE of the system results in k determining equations combined together and a system of linear partial differential equations (PDEs) on the coefficient functions ξ and η^i is then extracted. The final step is to solve this system of PDEs to find the coefficients $(\eta^i)'$ and $(\eta^i)''$.

Gödel's metric in a cylindrical coordinate system (t, r, ϕ, z), where $t < \infty$, $0 \leq r \leq \infty$, $0 \leq \phi \leq 2\pi$, $-\infty < z < \infty$, given by

$$ds^2 = a^2\left([dt + \sqrt{2}\sinh^2 r d\phi]^2 - dr^2 - dz^2 - \sinh^2 r \cosh^2 r d\phi^2\right), \tag{6}$$

ascertain the geodesic equations

$$\ddot{t} + \frac{4\sinh r}{\cosh r}\dot{t}\dot{r} + \frac{2\sqrt{2}\sinh^3 r}{\cosh r}\dot{r}\dot{\phi} = 0, \tag{7a}$$

$$\ddot{r} + 2\sqrt{2}\sinh r \cosh r\dot{t}\dot{\phi} - \sinh r \cosh r(1 - 2\sinh^2 r)\dot{\phi}^2 = 0, \tag{7b}$$

$$\ddot{\phi} - \frac{2\sqrt{2}}{\sinh r \cosh r}\dot{t}\dot{r} + \frac{2}{\sinh r \cosh r}\dot{r}\dot{\phi} = 0, \tag{7c}$$

$$\ddot{z} = 0. \tag{7d}$$

where dot over head the variables t, r, ϕ and z denote the derivatives with respect to the arc length parameter s. The solution of these equations have been a topic of interest to many researchers following different approaches. Chandrasekhar [7] used the classical integration whereas Novello et. al. [8] used the effective potential approach besides classical integration to solve them. Later Camci used dynamical symmetries [9] to find first integrals of these equations.

Here we find the Lie point symmetries of the system (7) and use them to reduce the order which then leads to a complete solution of the system. We developed a Maple procedure *symmetrygenerators* for finding the Lie point symmetries of an autonomous system. The inputs of the procedure are maximum of four autonomous ordinary differential equations in the coordinates t, x, y and z with the independent parameter s and the outputs are the coefficients ξ, η^1, η^2, η^3, η^4 of the symmetry generator. The main commands in the procedure are *Physics[diff], diff, coeffs, collect, subs, eval* and *pdsolve*.

This code is then applied for the system of geodesic equations of Gödel's metric and this in return gives ten Lie point symmetries, providing a basis of ten dimensional Lie algebra of generators:

$$\begin{aligned} X_1 &= s\frac{\partial}{\partial s},\ X_2 = z\frac{\partial}{\partial s},\ X_3 = \frac{\partial}{\partial s} \\ X_4 &= -\sqrt{2}\tanh r\sin\phi\frac{\partial}{\partial t} + \cos\phi\frac{\partial}{\partial r} - \frac{2\cosh^2 r - 1}{\sinh r\cosh r}\sin\phi\frac{\partial}{\partial \phi}, \\ X_5 &= \sqrt{2}\tanh r\cos\phi\frac{\partial}{\partial t} + \sin\phi\frac{\partial}{\partial r} + \frac{2\cosh^2 r - 1}{\sinh r\cosh r}\cos\phi\frac{\partial}{\partial \phi}, \\ X_6 &= \frac{\partial}{\partial \phi},\ X_7 = \frac{\partial}{\partial t},\ X_8 = s\frac{\partial}{\partial z},\ X_9 = z\frac{\partial}{\partial z},\ X_{10} = \frac{\partial}{\partial z}. \end{aligned} \tag{8}$$

The seven dynamical symmetries found in [9] are included in the above set. All Lie brackets are vanishing except

$$\begin{gathered} [X_1, X_2] = -X_2,\ \ [X_1, X_3] = -X_3,\ \ [X_1, X_8] = X_8,\ \ [X_2, X_8] = X_9 - X_1, \\ [X_2, X_9] = -X_2,\ \ [X_2, X_{10}] = -X_3,\ \ [X_3, X_8] = X_{10},\ \ [X_4, X_5] = 2\sqrt{2}X_7 + 4X_6, \\ [X_4, X_6] = X_5,\ \ [X_5, X_6] = X_4,\ \ [X_8, X_9] = X_8,\ \ [X_9, X_{10}] = -X_{10}. \end{gathered}$$

Accordingly, it includes a seven dimensional solvable sub-algebra and three-dimensional abelian sub-algebra.

It is known that a system of n kth-order ODEs $x_i^{(k)} = f_i(s, x, ..., x^{(k-1)})$, $i = 1, ..., n$ is solvable by quadratures if it admitts a kn-dimensional transitive solvable Lie sub-algebra[3]. This is not applicable in our case for any reduction of the order of the system since the derived algebra

$$L_{10} = L_{10}^{(1)}.$$

But we can profit from the commutative Lie sub-algebra in finding first integrals. Taking this in consideration, we find the solution of

$$Af = 0, \tag{9}$$

where A is the associated partial differential operator given by

$$A = \frac{\partial}{\partial s} + \dot{t}\frac{\partial}{\partial t} + \dot{r}\frac{\partial}{\partial r} + \dot{\phi}\frac{\partial}{\partial \phi} + \dot{z}\frac{\partial}{\partial z} + \ddot{t}\frac{\partial}{\partial \dot{t}} + \ddot{r}\frac{\partial}{\partial \dot{r}} + \ddot{\phi}\frac{\partial}{\partial \dot{\phi}} + \ddot{z}\frac{\partial}{\partial \dot{z}}, \tag{10}$$

which are found to be

$$c_1 = \sqrt{2}\dot{t}\sinh^2 r + \dot{\phi}\sinh^2 r(\sinh^2 r - 1), \tag{11}$$

$$c_2 = \dot{t} + \sqrt{2}\sinh^2 r\dot{\phi}, \tag{12}$$

$$\begin{aligned} c_3 &= \left(\dot{t} + \sqrt{2}\sinh^2 r\dot{\phi}\right)^2 - \sinh^2 r\cosh^2 r\dot{\phi}^2 - \dot{r}^2 \\ &= c_2\dot{t} + c_1\dot{\phi} - \dot{r}^2. \end{aligned} \tag{13}$$

Other well known procedure is given by the Cartan theory according to which, there exists a first integral, $X_a\dot{x}^a$ for each symmetry generator $X = \xi_a\partial_a$ obtained in eqs.(8) which satisfies the equations of Killing [2]

$$X_{a;b} + X_{b;a} = 0. \tag{14}$$

It is straight forward to check that the symmetry generators X_i where $i = 4..7$ and 10 satisfy eqs.(14). The corresponding first integrals are therefore (11), (12), (13) and

$$a = -\sinh r \cosh r \sin\phi \left[2\sqrt{2}\dot{t} + \dot{\phi}(2\sinh^2 r - 1)\right] - \dot{r}\cos\phi, \tag{15}$$

$$b = \sinh r \cosh r \cos\phi \left[2\sqrt{2}\dot{t} + \dot{\phi}(2\sinh^2 r - 1)\right] - \dot{r}\sin\phi. \tag{16}$$

(17)

The above equations give explicit expression of $\dot{x}^a$ reducing the system to

$$\dot{t} = c_2\left[1 - \frac{2\sinh^2 r}{\cosh^2 r}\right] + \frac{\sqrt{2}c_1}{\cosh^2 r}, \tag{18a}$$

$$\dot{\phi} = \frac{\sqrt{2}c_2}{\cosh^2 r} - \frac{c_1}{\sinh^2 r \cosh^2 r}, \tag{18b}$$

$$\dot{r}^2 = c_2^2 - c_3 - \left(\frac{\sqrt{2}c_2 \sinh r}{\cosh r} - \frac{c_1}{\sinh r \cosh r}\right)^2, \tag{18c}$$

$$\dot{r} = -\left(a\cos\phi + b\sin\phi\right). \tag{18d}$$

Then using the transformation $u = \sinh^2 r$, and integrating, provided that the tangent vector $\dot{x}^a$ and the associated underlying curve $x^a(s)$ are timelike, give the trajectories in the Gödel universe as

$$(t, r, \phi, z) = \left(\sqrt{2}\tan^{-1}\left(\sqrt{\frac{\alpha+1-\sqrt{\alpha^2-\beta^2}}{\alpha+1+\sqrt{\alpha^2-\beta^2}}}\tan\left(\sqrt{c_2^2+c_3}s + \frac{s_\circ}{2}\right)\right) - c_2 s + t_\circ,\right.$$

$$\sinh^{-1}\sqrt{\alpha + \sqrt{\alpha^2-\beta^2}\cos(\varepsilon s + s_\circ)},$$

$$\left.\tan^{-1}\left(\frac{\left(\sqrt{\frac{\alpha+1-\sqrt{\alpha^2-\beta^2}}{\alpha+1+\sqrt{\alpha^2-\beta^2}}} - \sqrt{\frac{\alpha-\sqrt{\alpha^2-\beta^2}}{\alpha+\sqrt{\alpha^2-\beta^2}}}\right)\tan\left(\sqrt{c_2^2+c_3}s + \frac{s_\circ}{2}\right)}{1 + \sqrt{\frac{\alpha+1-\sqrt{\alpha^2-\beta^2}}{\alpha+1+\sqrt{\alpha^2-\beta^2}}}\sqrt{\frac{\alpha-\sqrt{\alpha^2-\beta^2}}{\alpha+\sqrt{\alpha^2-\beta^2}}}\tan^2\left(\sqrt{c_2^2+c_3}s + \frac{s_\circ}{2}\right)}\right) + \phi_\circ, -c_\circ z + z_\circ\right). \tag{19}$$

The graphs of the trajectories in (r, ϕ)-plane for all possible values of the parameters c_1, c_2 and c_3 appearing in the solution are given below.

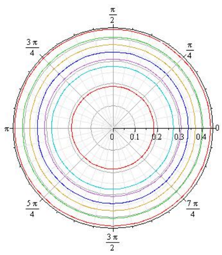

Figure 1: The graphs of $0 < u(s) < \dfrac{-1+\sqrt{2}}{2}$ when $c_1 = c_{1min}$.

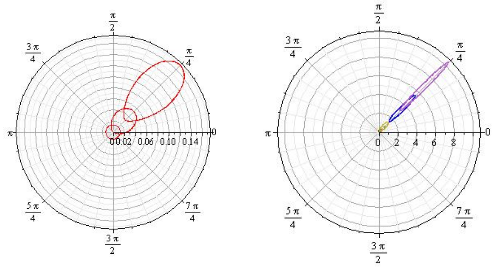

Figure 2: Trajectories in (y, ϕ) plane with increasing c_1, and c_2, c_3 are fixed

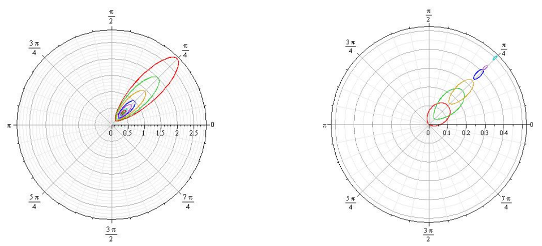

Figure 3: (a)Trajectories with c_1 and c_2 are fixed, $0 < c_3 < c_2^2$. (b)Trajectories with c_1 is increasing as $c_3 \to c_2^2$.

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