Homogeneous AdS Black Strings and Black p-branes in Einstein gravity and pure Lovelock-Maxwell theory

In this work we introduce a new set of solutions of General Relativity with a negative cosmological constant, that describe black strings and black p-branes. The solutions are supported by p minimally coupled, free scalars, that have a linear dependence on the Cartesian coordinates along the extended flat directions, having therefore a finite energy density. This scalar field provide a momentum dissipation phenomena that allows to describe holographic dual with finite conductivities. We extend our result to pure Lovelock-Maxwell theory.

Introduction. — It has been well established by now that general relativity in spacetime dimensions greater than four admits black object solutions with event horizons of different topologies [1]; the archetypical example being the black ring [2, 3]. The black ring is a Ricci flat black hole whose event horizon has topology $S^2 \times S^1$, in contrast to the S^3 topology of the Myers-Perry generalization of Kerr geometry [4]. The existence of such solutions shows how, in five or higher dimensions, the theory circumvents topological obstructions that in four dimensions it encountered for admitting hairy solutions in asymptotically flat space [5].

For large angular momentum, the black ring solution can be described by a black string geometry, which can be thought of adding an unwrapped flat direction to the four-dimensional Schwarzschild solution. By warping the four-dimensional Schwarzschild-AdS solution and adding an extra dimension to it, one can easily construct analytic black strings in AdS space. The warping factor, however, makes the AdS black string to be non-uniform, and this introduces difficulties relative to the asymptotically flat case, specially in relation to the study of its dynamical stability as well as a proper definition of energy density. Relying on numerical tools, homogenous black strings in AdS can be constructed in pure GR with a negative cosmological constant [6], [7] as well as in five dimensional gauged supergravity [8]. In this paper, we prove that general relativity with negative cosmological constant, apart from admitting such warped-AdS and numerical black string solutions, also admits analytic solutions that describe homogeneous black strings and black p-branes. These solutions are supported by minimally coupled, free scalar fields and exist in arbitrary dimension D greater than 4.

New black strings in AdS. — Consider Einstein theory in dimension $D = d + p$, coupled to p scalar fields $\psi^{(i)}$ with $i = 1, 2, ..., p$. The field equations are given by

$$G_{AB} + \Lambda g_{AB} = \kappa \sum_{i=1}^{p} T^{(i)}_{AB} \ , \qquad (1)$$

with

$$T^{(i)}_{AB} = \frac{1}{2}\partial_A \psi^{(i)} \partial_B \psi^{(i)} - \frac{1}{4} g_{AB} \partial_C \psi^{(i)} \partial^C \psi^{(i)} \ , \qquad (2)$$

and

$$\Box \psi^{(i)} = 0 \text{ with } i = 1, 2, ...p \ . \qquad (3)$$

Here, G_{AB} is the Einstein tensor. Hereafter, we will set $\kappa = 16\pi G = 1$.

The theory defined above admits the following solution

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2 d\Omega^2_{d-2,\gamma} + \delta_{ij} dx^i dx^j \qquad (4)$$

provided

$$F(r) = \gamma - \frac{2\mu}{r^{d-3}} - \frac{2\Lambda r^2}{(d-1)(d+p-2)} \ , \qquad (5)$$

with x^i $(i = 1, ..., p)$ being Cartesian coordinates. These are the coordinates along the flat $p-$brane. Remarkably, the solution for the fields takes the simple form

$$\psi^{(i)} = \lambda x^i \ , \qquad (6)$$

with

$$\lambda^2 = -\frac{4\Lambda}{(d+p-2)} \ . \qquad (7)$$

That is, the scalar fields have a linear dependence with the coordinates x^i. In (4)-(5), μ appears as an arbitrary integration constant, and $\gamma = \pm 1, 0$ is the curvature of an Euclidean manifold of constant curvature of dimension $d-2$ and line element $d\Omega^2$. Note that (7) forces the *bare* cosmological constant Λ to take a negative value.

The solution presented above is the first of its type of having been found analytically; namely, it is the first homogeneous, analytic black p-brane solution of Einstein equations with non-vanishing cosmological constant.

Spacetime (4) is asymptotically $AdS_d \times R^p$, with the curvature radius of the AdS_d factor given by

$$\frac{1}{l^2} = -\frac{2\Lambda}{(d-1)(d+p-2)} = -\frac{2\Lambda}{(D-p-1)(D-2)} \ . \qquad (8)$$

Notice that this value for the *dressed* AdS_d curvature radius l, obtained from (8), differs from the *bared* value of the maximally symmetric AdS_D solution of the theory, $l_0^{-2} = -2\Lambda / [(D-1)(D-2)]$. In general, $l \leq l_0$, with the upper bound corresponding to $p = 0$.

We can choose $\gamma = \pm 1, 0$, which leads to three possible local geometries for the asymptotic boundary. That is to say, the holographic dual field theory can in principle be formulated either on R^{D-1} for $\gamma = 0$, $R \times S^{d-1} \times R^p$ for $\gamma = 1$, or $R \times H^{d-1} \times R^p$ for $k = -1$.

General Construction. — Let us now consider a general D-dimensional metric of the form

$$ds_D^2 = d\tilde{s}_d^2 + \delta_{ij} dx^i dx^j \ , \tag{9}$$

and the set of scalar fields $\psi^{(i)} = \lambda x^i$, where we have split the indices in such a way that Greek indices and tilded objects live on the manifold with line element $d\tilde{s}$, while lowercase Latin indices run along the p extended directions.

Einstein equations (1) projected along the manifold $d\tilde{s}$ and the extended directions x^i, respectively reduce to

$$\tilde{G}_{\mu\nu} + \left(\Lambda + \frac{p\lambda^2}{4}\right) \tilde{g}_{\mu\nu} = 0 \ , \tag{10}$$

and

$$\tilde{R} = 2\Lambda - \left(1 - \frac{p}{2}\right) \lambda^2 \ . \tag{11}$$

The compatibility of the trace of (10) –obtained by contracting such equation with $\tilde{g}^{\mu\nu}$– with equation (11) implies that the constant λ must be fixed as in (7). In other terms, the configuration of the scalar fields induces a shift in the cosmological constant of any d-dimensional Einstein manifold. Therefore, on the transverse section of the p-brane we can consider any solution to Einstein equation in d dimensions, provided (10) is obeyed. We can, for example, consider the asymptotically AdS rotating solution of general relativity with negative cosmological constants, which is characterized by its mass as well as $\left[\frac{d-2}{2}\right]$ angular momenta [9–11], to construct black strings in $AdS_d \times R^p$, with a rotating black hole on the brane. It is important to stress that these compatibility relations cannot be satisfied in general relativity when electric charges are included.

Charged homogeneous black strings in AdS. — In order to extend our solutions to include electric charge we follow what has been recently found in [12]. In [12] it has been shown how to construct analytic black string solution in the presence of p-forms using higher curvature gravity theories. In here we extend those results to construct homogeneous AdS black strings and black p-branes. Let us considere the following theory

$$\mathcal{I} = \int \sqrt{-g} d^D x \left[\mathcal{L}^n + 2\Lambda - \frac{1}{2p} F_{\alpha 1..\alpha p} F^{\alpha 1..\alpha p}\right] \tag{12}$$

$$-\frac{1}{2} \sum_{i=D-q}^{q} \int \sqrt{-g} d^D x (\partial\psi_i)^2$$

where $\mathcal{L}^n$ is the term of order n in the Lovelock series

$$\mathcal{L}^n = \frac{1}{2^n} \delta^{C_1 D_1 \ldots C_n D_n}_{A_1 B_1 \ldots A_n B_n} R^{A_n B_n}_{C_n D_n},$$

$F_{\alpha 1..\alpha p}$ is the field strength of a $(p-1) - form$ and ψ_i are q minimally coupled free scalar fields that depends only on the extended flat directions x_q. We will show that for the case of $\mathcal{L}^2$ it is possible to construct AdS charged black strings and black p-branes (according to the number of extended flat directions) making use of our linearly dependent scalar fields.

Further comments. — The black p-brane solutions are supported by the scalar fields $\psi^{(i)}$, which are linear on the coordinates x^i. Even though these fields diverge in the limit $x^i \to \pm\infty$, they yield finite energy density. That is, the divergence merely comes from the non-compactness of the extended directions. In fact, the tt component of the energy-momentum tensor for the collection of $\psi^{(i)}$ turns out to be independent of the coordinates x^i. Therefore, one can properly define the energy density, as in the Ricci-flat, homogenous black strings.

Solution (4)-(6) exists due to the fact that, despite the metric being homogenous, the scalar fields break translational symmetry. This idea has been used in many different contexts, for example in the construction of boson stars and other gravitational solitons [13, 14] as well as for rotating hairy black holes [15].

Each of the scalars in the solution can be dualized to $(D-1)$-forms. Both the formulations in terms of scalars and in terms of the $(D-1)$-forms have resulted very useful to construct asymptotically AdS, planar black holes in the presence of other matter fields, which is of great importance in the context of AdS/CFT correspondence and, in particular, to its applications to condensed matter (see e.g. [16, 17]).

The entropy density is given by the area law, namely

$$s = \frac{S}{V} = \frac{r_+^{d-2}\sigma}{4G} = 4\pi r_+^{d-2}\sigma \ , \tag{13}$$

where σ is the unit volume of $\Omega_{d-2,\gamma}$, V is the volume of the extended directions and r_+ is the largest root of the equation $0 = F(r_+) := -g_{tt}$. The temperature, T, can be obtained as usual from the period of the Euclidean time that yields a regular Euclidean section of the imaginary continuation $t \to it$. This yields,

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{4\pi}\left(\frac{(d-3)}{r_+} - \frac{2\Lambda}{d+p-2} r_+\right). \tag{14}$$

Since there is no charge associated to the free scalar fields in this family of solutions, the mass M –or the energy density $m = M/V$– can directly be obtained from the first law of black hole mechanics; namely $dm = Tds$. Therefore,

$$m = \sigma (d-2) r_+^{d-3} \left(1 - \frac{2\Lambda}{(d-1)(d+p-2)} r_+^2\right) \ . \tag{15}$$

Before concluding, let us mention that black strings suffer from Gregory-Laflamme (GL) instabilities [18], namely long-wavelength, perturbative instability triggered by a mode travelling along the extended directions.

This kind of instability goes beyond the realm of general relativity, and pervades also black string solutions in other theories, like higher-curvature gravities [19–23]. Numerical simulations show that in five dimensions the GL instability leads to the formation of a naked singularity [24, 25], while thermodynamical arguments indicate that for dimensions greater than 13 the final stage of the instability could be an inhomogeneous black string [26]. The latter has been recently confirmed in the large D limit [27]. It is likely that small (as compared with l) black strings will suffer from a GL instability. Whether or not that is the case, is beyond the scope of the present work and is matter of our current research. One might nevertheless expect, in the context of the AdS/CFT duality [28]-[29], that if the dual CFT has a well defined evolution, the existence of an instability shouldn't lead to a naked singularity in the bulk as in the Ricci flat case. For the case of our charged black strings solutions we will show that in order to get homogenous AdS solutions the inclusion of the scalar fields is fundamental, and that they are provided by the following relation between the axion charge, the cosmological constant and the number of extended flat directions

$$\lambda^2 = -\frac{16\Lambda}{2(d+q)-8} \tag{16}$$

where $D = d + q$ being q the flat extended directions.

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