Thermodynamics sheds light on black hole dynamics

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I. INTRODUCTION

Thermodynamics has proven to be a powerful tool to give a physical interpretation of the integration constants of Einstein's equations characterizing a black hole spacetime, in introducing its extensive parameters (global charges and entropy) and its intensive ones defined on the horizon (temperature, electric potential, etc.). The first law then tells us how a black hole readjusts its equilibrium configuration when interacting with its environment.

On the other hand, the dynamics of interacting (non rotating) compact objects relies on reducing them to point particles endowed with an effective mass parameter. In the case of a general relativistic Schwarzschild black hole, the interpretation of this parameter seems straightforward since it identifies to the Schwarzschild mass, which is the only integration constant at hand.

Consider now as an example Einstein-Maxwell-dilaton theories, which consist in supplementing general relativity with a scalar field and a (non-minimally coupled) vector field. Such theories allow for the existence of hairy black hole solutions depending no longer on one, but on three integration constants. Their reduction to point particles was recently performed in [1] and involves a scalar-field-sensitive mass $m(\varphi)$ a la Eardley [2]. The explicit calculation of this black hole "sensitivity" includes a constant parameter μ which identifies to the Schwarzschild mass when the hairs are cut off.

We will show that this constant μ must be defined in terms of the entropy of the black hole alone. This can be understood thus: Eardley's $m(\varphi)$ modeling means that the black hole is moving adiabatically in the fields of its companion; this will imply that it satisfies the first law of thermodynamics; moreover, the specific form of $m(\varphi)$ previously obtained in [1] will impose that it exchanges no mass nor charge with its environment. Therefore, its entropy will remain constant and hence, can be related to μ.

Returning then to the general relativistic Schwarzschild black hole, this result shows that the constant parameter describing it as a point particle must not be interpreted as its mass but rather be related to its entropy. Since the second law tells us that the entropy must increase, this means that, at a better, non adiabatic approximation, its effective mass can no longer be taken as a constant.

II. THE EXAMPLE OF EMD BLACK HOLES AND THEIR REDUCTION TO POINT PARTICLES

The vacuum Einstein-Maxwell-dilaton action of gravity is taken to be, see [3–5]:

$$16\pi\, I[g_{\mu\nu}, A_\mu, \varphi] = \int d^4x\sqrt{-g}\left(R - 2g^{\mu\nu}\partial_\mu\varphi\,\partial_\nu\varphi - e^{-2a\varphi}F^2\right)\ , \tag{II.1}$$

where g is the determinant of the metric $g_{\mu\nu}$, R is the Ricci scalar, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with $F^2 = F^{\mu\nu}F_{\mu\nu}$, and where a parametrizes the theory.

The field equations derived from the action (II.1) are :

$$R_{\mu\nu} = 2\partial_\mu\varphi\,\partial_\nu\varphi + 2e^{-2a\varphi}\left(F_\mu{}^\lambda F_{\nu\lambda} - \frac{1}{4}g_{\mu\nu}F^2\right), \tag{II.2a}$$

$$D_\mu\left(e^{-2a\varphi}F^{\mu\nu}\right) = 0 \quad \text{and} \quad \Box\,\varphi = -\frac{a}{2}e^{-2a\varphi}F^2. \tag{II.2b}$$

The "electrically" charged, static, spherically symmetric black hole solutions of the equations above which will best illustrate the correspondence between their thermodynamics and dynamics were found in [3–5], and read, with

$d\Omega^2 \equiv d\theta^2 + \sin^2\theta\, d\phi^2$,

$$ds^2 = -\left(1-\frac{r_+}{r}\right)\left(1-\frac{r_-}{r}\right)^{\frac{1-a^2}{1+a^2}} dt^2 + \left(1-\frac{r_+}{r}\right)^{-1}\left(1-\frac{r_-}{r}\right)^{-\frac{1-a^2}{1+a^2}} dr^2 + r^2\left(1-\frac{r_-}{r}\right)^{\frac{2a^2}{1+a^2}} d\Omega^2 \ , \tag{II.3}$$
$$A_t = -\sqrt{\frac{r_+ r_-}{1+a^2}}\,\frac{e^{a\varphi_\infty}}{r} \ , \qquad A_i = 0 \ , \qquad \varphi = \varphi_\infty + \frac{a}{1+a^2}\ln\left(1-\frac{r_-}{r}\right) \ .$$

In order to be able to address the post-Newtonian dynamics of two interacting EMD black holes, we have to explicitely include them as sources in the action. They were phenomenologically replaced in [1] by point particles described by the following "skeleton" action:

$$I^{\rm pp}[g_{\mu\nu}, A_\mu, \varphi, \{x^\mu_A\}] = I - \sum_A \int m_A(\varphi) ds_A + \sum_A q_A \int A_\mu\, dx^\mu_A \, , \tag{II.4}$$

where I is given in (II.1), $ds_A = \sqrt{-g_{\mu\nu}dx^\mu_A dx^\nu_A}$, and where $x^\mu_A[s_A]$ is the worldline of the skeletonized black hole A. The parameter q_A is taken to be a constant in order to preserve the $U(1)$ symmetry of the full action. As for the scalar-field-sensitive, effective, "mass" function $m_A(\varphi)$ it is taken to be a function of the value of the scalar field at the location of the particle A. The calculation of such mass functions is standard when the compact object is a neutron star, see e.g. [6] and [7]. Let us recall here briefly how it was, for the first time, computed in [1] when the compact object A is the EMD black hole described above.

The field equations derived from (II.4) are the same as (II.2) but supplemented by point source terms. They were solved in [1] in the near-worldline region of the particle A, in its rest frame, and at linear order around a background solution consisting of an asymptotically flat spacetime, a vector field which can be "gauged away" to zero, and an asymptotic scalar field environment φ_∞ that is imposed by the faraway companion B. The solutions were then identified to the EMD black hole solution (II.3) at leading, $\mathcal{O}(1/r)$, order to yield (dropping, from now on, the index A):

$$q = \sqrt{\frac{r_+ r_-}{1+a^2}}\, e^{-a\varphi_\infty} \ , \tag{II.5a}$$
$$m(\varphi_\infty) = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2}\, r_-\right) \ , \tag{II.5b}$$
$$\frac{dm}{d\varphi}(\varphi_\infty) = \frac{a\, r_-}{1+a^2} \ . \tag{II.5c}$$

The system (II.5) is integrable : indeed, expressing r_+ and r_- in terms of m and $dm/d\varphi$ using (II.5b) and (II.5c), and injecting the result into (II.5a) gives the first order differential equation

$$\left[\left(\frac{dm}{d\varphi}\right)\left(m(\varphi) - \frac{1-a^2}{2a}\frac{dm}{d\varphi}\right)\right]_{\varphi=\varphi_\infty} = \frac{a}{2}\, q^2 e^{2a\varphi_\infty} \ . \tag{II.6}$$

This differential equation can be solved for all a, see [1]. The solution reads as $F[m(\varphi_\infty),\, q,\, \varphi_\infty,\, a] = \mu^2$ where the explicit form of F can be found using e.g. Mathematica and where μ is an integration constant. In the case $a = 1$, which is enough to illustrate our purposes, the solution simply is

$$m(\varphi) = \sqrt{\mu^2 + q^2\frac{e^{2\varphi}}{2}} \ , \tag{II.7}$$

where the index ∞ has been dropped since the scalar background φ_∞, imposed by B, can have any value. One can then address the post-Newtonian dynamics of the two black holes and, for example, compute the two-body PN Lagrangian, see [1].

A question left pending at this stage is the relationship between the constants q and μ characterizing the skeletonized black hole and its extensive parameters, that is, its electric charge Q, mass M and entropy S.

III. THERMODYNAMICS VERSUS DYNAMICS OF EMD BLACK HOLES

The first law of thermodynamics obeyed by EMD black holes is found in the standard way:
Their temperature T is defined as

$$T \equiv \frac{\kappa}{2\pi} = \frac{1}{4\pi r_+}\left(1-\frac{r_-}{r_+}\right)^{\frac{1-a^2}{1+a^2}} \ , \tag{III.1}$$

where κ is their surface gravity, with $\kappa^2 = -\frac{1}{2}(\nabla_\mu \xi_\nu \nabla^\mu \xi^\nu)_{r_+}$, and $\xi^\mu = (1,0,0,0)$ being the timelike Killing vector. Their electric potential is

$$\Phi \equiv A_t(r \to \infty) - A_t(r_+) = \sqrt{\frac{r_-}{(1+a^2)r_+}} e^{a\varphi_\infty} \ . \tag{III.2}$$

The action for the metric being Einstein-Hilbert's, the entropy S of the black holes is the fourth of their horizon area A_+:

$$S \equiv \frac{A_+}{4} = \pi r_+^2 \left(1 - \frac{r_-}{r_+}\right)^{\frac{2a^2}{1+a^2}} \ . \tag{III.3}$$

As for the global charges associated to these solutions, that is, their electric charge Q and mass M, they can be obtained within various approaches, e.g. the Hamiltonian one as developped by Regge-Teitelboim [8] or the Lagrangian one as developped by Katz [9, 10]. As usual in scalar-tensor theories of gravity, the scalar field contributes to the on-shell Hamiltonian at infinity so that

$$Q = \sqrt{\frac{r_+ r_-}{1+a^2}}\, e^{-a\varphi_\infty} \ , \tag{III.4a}$$

$$M = \frac{1}{2}\left(r_+ + \frac{1-a^2}{1+a^2} r_-\right) - \frac{a}{1+a^2}\int r_- d\varphi_\infty \ . \tag{III.4b}$$

As one can see, M is the sum of the ADM mass (which identifies to one-half the $\mathcal{O}(1/r)$ coefficient of g_{rr} at spatial infinity) and of a scalar contribution [11–13] (which is called the scalar charge in [14]) see, e.g., [15, 16].
With all these definitions in hand, it is easily checked that the variations of S, Q, and M with respect to r_+, r_- and φ_∞, are such that the following identity holds (which is an equivalent rewriting of the first law in [14]):

$$T\delta S = \delta M - \Phi \delta Q \, . \tag{III.5}$$

We show now how the first law of thermodynamics (III.5) *justifies* the skeletonization of EMD black holes introduced in section II and provides an interpretation of the constants q and μ that characterize it.

Comparing (III.4a) and (II.5a) we first see that we must identify the constant q, called q_A in the skeleton action (II.4), to the global electric charge Q of the black hole. The significance of this identification is that the dynamical evolution of the skeletonized black hole is such that its charge remains constant, $\delta Q = 0$.

Second, the variation of the global black hole mass M, when interacting with its environment, follows from (III.4b) and reads

$$\delta M = \frac{1}{2}\delta\left(r_+ + \frac{1-a^2}{1+a^2} r_-\right) - \frac{a\, r_-}{1+a^2}\,\delta\varphi_\infty \ , \tag{III.6}$$

which is zero when taking into account (II.5b) and (II.5c). This means that the black hole global mass remains constant as well during its dynamical evolution, $\delta M = 0$ (while the ADM mass and "scalar charge" are not separately conserved).

Therefore, the skeletonization of black holes proposed in (II.4) amounts to describing them as remaining isolated when, for example, they orbit around a companion.

Finally, the first law (III.5) tells us that, since $\delta Q = 0$ and $\delta M = 0$, the entropy of the black hole remains constant as well : $S = const$. This is the main result of this paper, which shows that when one reduces a black hole to a point particle a la Eardley (II.4), one in fact describes a black hole whose equilibrium configuration readjusts adiabatically when interacting with its companion, such that is mass M, charge Q, and hence entropy S remain constant.

Therefore, it must be possible to define the parameter μ appearing in the "mass" function $m(\varphi)$ when integrating (II.6) as a function of the entropy S only (or, equivalently, the "irreducible mass" $M_{\rm irr}$ [17, 18]). That this is so can be shown for all a: indeed, inserting the expressions (II.5a) and (II.5b) for q and $m(\varphi)$ in the solution $F[m(\varphi), q, \varphi, a] = \mu^2$, one finds that μ^2 identifies with $S/4\pi \equiv M_{\rm irr}^2$ as given in (III.3). Since, moreover, the parameter q must be identified to the graviphoton charge Q, the solution $F[m(\varphi), Q, \varphi, a] = S/4\pi$ implicitely gives $m(\varphi)$ in terms of Q, S, and φ. In the illustrative example $a = 1$ where $m(\varphi)$ is given in (II.7), this yields:

$$\mu^2 = \frac{S}{4\pi} \quad \text{so that} \quad m(\varphi) = \sqrt{\frac{S}{4\pi} + Q^2 \frac{e^{2\varphi}}{2}} \, . \tag{III.7}$$

Note that when $r_- = 0$ ($\forall a$) the black hole solution is Schwarzschild's so that $m(\varphi)$ is reduced, as it should, to its (constant) mass $m = \sqrt{S/4\pi} = r_+/2$. The same is true in the Reissner-Nordström limit, $a = 0$, for which $m = (r_+ + r_-)/2$, see (II.5). On the other hand, when a non-trivial scalar field is present, the phenomenological, "Eardley-inspired", scalar-field-sensitive mass function $m(\varphi)$ for black holes which was shown in [1] to satisfy (II.6) is in fact explained and justified by their thermodynamics, and the parameters q and μ become related to their global electric charge and *entropy*.

IV. CONCLUSION

The results above indicate that the conservative dynamics of a (hairy) black hole when skeletonized "à la" Eardley, as in (II.4), is generically such that it does not exchange energy (nor electric charge) with its environment. Therefore, because of the first law of thermodynamics, the black hole adiabatically readjusts its equilibrium configuration in such a way that its entropy (or area in the case at hand) remains constant. We conjecture that this fact holds in any scalar-vector-tensor theory of gravity and that the parameters entering the scalar-field-sensitive "mass" functions attributed to skeletonized black holes can always be related to their global gauge charges and their Wald entropy [19], which remain constant in their motion around their companion.

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