# A holographic approach to gravitational thermodynamics

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We formulate a description of 3+1 dimensional gravitational phenomena in terms of a relativistic fluid living on the 2+1 dimensional time-like boundary of an arbitrary bulk region of space-time, called a *gravitational screen*. We establish a consistent dictionary between the geometric variables describing the evolution of the screen and the thermodynamics variables describing a relativistic viscous fluid, and discuss the interpretation. We also examine the construction of gravitational screens in different spacetimes and analyze the properties of the fluids they realize.

*Keywords*: Gravity; thermodynamics; quasi-local; holography; hydrodynamics.

## 1. Introduction

One of the greatest theoretical developments in modern physics has been the anti-de Sitter/conformal field theory correspondence (AdS/CFT), which conjectures an equivalence between a theory of gravity in a bulk region of space-time, and a quantum field theory on the boundary of that space-time. AdS/CFT has illuminated many aspects of string theory as well as field theory, giving key insights into what a quantum theory of gravity might look like. More than that, it has provided us with the tools to study a broad range of strongly coupled systems, such as fluids near quantum critical points and quark-gluon plasmas. An intricately related concept is the membrane paradigm, which asserts that one can replace the interior of a black hole with a relativistic fluid living on its event horizon, and that the two would be indistinguishable to an outside observer. These ideas are all, in essence, a statement of the holographic principle; the fundamental idea that all of the information contained in a bulk region of space-time can be encoded in the boundary of that region.

These approaches offer a limited perspective however, since one is usually constrained to situations where knowledge of the boundary of space or the end of time is required. AdS/CFT makes reference to the boundary of anti-de Sitter space at infinity, while the membrane paradigm and standard formulations of black hole thermodynamics both define quantities on the event horizon, which is teleological and can only be located by knowing the entire future history of the universe. From a practical point of view this is unsatisfactory, since as local observers we are generally unable to access these types of boundaries.

Recent developments in addressing these issues has led to the concept of using a 'gravitational screen' as a quasi-local observer [1]. A gravitational screen is a 2+1 dimensional time-like hypersurface representing the time evolution of the 2d boundary of a region of space-time. Projecting Einstein's equations onto the screen leads to the equations governing the non-equilibrium thermodynamics of a viscous fluid, and encodes in the fluid all of the gravitational dynamics present inside the screen. This approach is reminiscent of the membrane paradigm[3] and fluid/gravity correspondence[4] but allows one to discuss thermodynamics on any time-like surface.

In this work, we present the fully relativistic generalization of the work done in Ref. 1, as well as correcting the interpretation of the fluid pressure as the screen's normal acceleration. We also construct examples of gravitational screens both in Minkowski and Schwarzschild backgrounds and examine the properties of the corresponding holographic fluids. We examine how the fluid entropy is linked to the curvature of spacetime, and remark on the salient features of the correspondence.

## 2. Screens as hypersurfaces

The gravitational screen $\Sigma$ is a 2+1 dimensional hypersurface which is the time evolution of a 2d boundary $\mathcal{S}$ of some 3d bulk region of interest. We consider spherically symmetric boundaries only, though the screen geometry can in principle be chosen arbitrarily. The screen is defined by $s^a$, the outward pointing space-like unit normal vector to $\mathcal{S}$, and $u^a$, the time-like unit vector tangent to $\Sigma$. By construction $s_a s^a = 1$, $u_a u^a = -1$, and $s_a u^a = 0$.

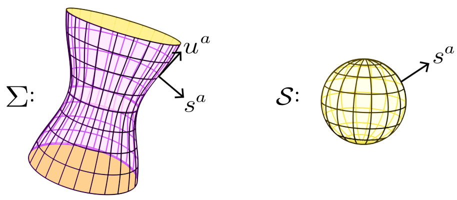


Fig. 1. A gravitational screen $\mathcal{S}$ and its time evolution $\Sigma$.

The metrics on $\Sigma$ and $\mathcal{S}$ in terms of the bulk metric $g_{ab}$ are

$$\Sigma : \quad h_{ab} = g_{ab} - s_a s_b \qquad\qquad \mathcal{S} : \quad q_{ab} = h_{ab} + u_a u_b \tag{1}$$

with the associated extrinsic curvatures $H_{ab} = h_a{}^c h_b{}^d \nabla_c s_d$ and $\Theta_{uab} \equiv q_a{}^c q_b{}^d \nabla_c u_d$. The junction conditions[6] then lead to a surface stress-energy tensor:

$$\tilde{S}_{ab} = \frac{1}{8\pi G} S_{ab} = \frac{1}{8\pi G}\big([H]h_{ab} - [H_{ab}]\big) \ , \qquad H \equiv h^{ab} H_{ab} \tag{2}$$

Square brackets represent the discontinuity of a quantity across the boundary. We will interpret this as the stress-energy tensor of a relativistic fluid living on the surface $\Sigma$. We adopt a holographic point of view, where the fluid stress-energy tensor (supported entirely on $\Sigma$) and the equations governing its evolution map to the gravitational dynamics within the screen. To this end, we let $[H_{ab}] \to H_{ab}$ and $[H] \to H$ giving

$$S_{ab} = H h_{ab} - H_{ab}. \tag{3}$$

This is in the same spirit as the membrane paradigm[3], where the stretched horizon is taken to be the boundary of the space-time and the stress energy tensor on the surface is chosen so that field lines terminate at the boundary.

To arrive at a precise correspondence, we take Einstein's equations in the 4d space with zero cosmological constant, $R - \frac{1}{2}Rg_{ab} = T_{ab}$, and project them onto the time-like membrane $\Sigma$ a la the Gauss-Codazzi equations, the first of which is the momentum constraint

$$D_b S^{ba} = T_{cb} s^c h^{ba} \tag{4}$$

where $D_a V_b = h_a{}^c h_b{}^d \nabla_c V_d$ is the covariant derivative on $\Sigma$. Equation (4) can further be projected into the spatial direction as $(D_b S^{ba}) q_{ac} = T_{sc}$, where $T_{sc} \equiv s^a T_{ac}$ and it is understood that the index $c$ represents components tangent to $S$. This equation expresses conservation of momentum on $\Sigma$, with $T_{sc}$ representing the momentum flux density across the screen. Using the surface stress-energy tensor (3) and inserting the factor $8\pi G$ gives

$$-(8\pi G)T_{sc} = \theta_s \mathrm{a}_{uc} + (\tfrac{3}{2}\theta_u q_{bc} + \tilde{\Theta}_{ubc} + \epsilon_{bc})\omega^b + D_u \omega_c - D_c(\gamma_u + \tfrac{1}{2}\theta_s) + D_b \tilde{\Theta}^b_{sc} \tag{5}$$

Here, $\mathrm{a}_{uc} \equiv u^b \nabla_b u_c$ is the acceleration of screen observers, $\omega_a \equiv q_a{}^b(s_c \nabla_b u^c)$ is the normal one-form, $\gamma_u \equiv s_b u^a \nabla_a u^b$ is the normal acceleration, $\tilde{\Theta}_{ubc}$ is the symmetric trace-free part of $\Theta_{ubc}$, $\epsilon_{bc}$ is the antisymmetric part of $\Theta_{ubc}$, which vanishes if $u^a$ is hypersurface orthogonal, and $\theta$ is the expansion. Projecting (4) in the direction of $u^a$ instead gives conservation of energy on the screen

$$-(8\pi G)T_{su} = -(D_u + \theta_s)\theta_u + (\gamma_u + \tfrac{1}{2}\theta_s)\theta_u + \tilde{\Theta}^{ab}_s \Theta_{sab} - (\mathrm{d}_a + 2\mathrm{a}_{ua})\omega^a \tag{6}$$

where $T_{su} \equiv s^a u^b T_{ab}$ represents the energy flux density flowing across the screen and we have defined the covariant derivative on $\mathcal{S}$ as $\mathrm{d}_a V_b = q^c{}_a q^b{}_d \nabla_c V_d$.

## 3. Hydrodynamics

We consider now the equations governing a relativistic viscous fluid living on a 2+1 dimensional surface $\mathcal{S}^2 \times \mathcal{R}$. The conservation equations are

$$\nabla_a T^{ab} = 0 \tag{7}$$

where in the general case we consider non-perfect (non-equilbrium) fluids with stress-energy tensors of the form

$$T^{ab} = e u^a u^b + (p + \pi) q^{ab} + u^a q^b + u^b q^a + \Pi^{ab}. \tag{8}$$

$u^a$ is the fluid 4-velocity, $e$ is the internal energy density of the fluid, $p$ is the isotropic pressure, $q^{ab}$ is the metric on $\mathcal{S}^2$, $\Pi^{ab}$ is the anisotropic stress tensor, $q^a$ is the heat flux, and $\pi$ is the dynamic pressure (the difference between the total pressure and pressure at equilibrium). $\Pi^{ab}$, $q^a$, and $\pi$ are thermodynamic fluxes that capture the deviations from a perfect fluid, and are given in this case by:

$$\Pi^{ab} = q^a{}_c q^b{}_d T^{cd} - (p+\pi)q^{ab} \ , \qquad q^a = -q^a{}_c T^{cb} u_b \ , \qquad p + \pi = \frac{1}{2} q_{ab} T^{ab} \tag{9}$$

As in the gravity picture, we project (7) in the directions parallel and orthogonal to $u^a$, giving the analogue of the Gauss-Codazzi equations but with $T_{ab}$ playing the role of $S_{ab}$. Projecting in directions orthogonal to $u^a$ gives conservation of momentum:

$$0 = \dot{u}_c(e + p + \pi) + \mathrm{d}_c(p+\pi) + (\mathrm{d}_a + \dot{u}_a)\Pi^a{}_c + q_{cb}\dot{q}^b + \left(\omega_{ac} + \sigma_{ac} + \tfrac{3}{2}\theta q_{ac}\right) q^a \tag{10}$$

and projecting in the direction of $u^a$ gives conservation of energy

$$0 = -\dot{e} - (e + p + \pi)\theta - (\mathrm{d}_a + 2\dot{u}_a)q^a + \Pi^{ab}\sigma_{ab} \tag{11}$$

## 4. The dictionary

We can now construct the mapping between the geometric variables describing the boundary evolution, and the thermodynamic variables governing the evolution of a relativistic fluid. Remarkably, this can be done consistently in the relativistic case just as in the Newtonian case considered in Ref. 1. Comparing the conservation of momentum (10) and energy (11) equations for the fluid to the gravity equations (5) and (6) leads to the following identifications:

$$e = -\frac{\theta_s}{8\pi G} \qquad p + \pi = \frac{\gamma_u + \frac{1}{2}\theta_s}{8\pi G} \qquad \theta = \theta_u \qquad \omega_{ab} = \epsilon_{ab} \tag{12}$$

$$\Pi_{ab} = -\frac{\tilde{\Theta}_{sab}}{8\pi G} \qquad \sigma_{ab} = \Theta_{uab} \qquad q_a = -\frac{\omega_a}{8\pi G} \tag{13}$$

We initially took the hydrodynamic conservation laws (7) to be source-free, resulting in the vanishing left hand side of (10) and (11). It is clear now that a non-zero $T_{su}$ or $T_{sc}$, which represent respectively the energy and momentum flux across the screen, manifest themselves as non-zero source terms in (7).

In this picture, the fluid energy density $e$ is related to the rate of expansion of outgoing radial null geodesics at the boundary. Our choice of sign leads to a negative energy density which increases with the mass of the screened region. Positivity of entropy and temperature for the screen fluid requires the presence of a negative chemical potential $\mu$ for the fluid. The 2d fluid pressure $p$ is now identified with $\gamma_u + \frac{1}{2}\theta_s$, in contrast with both the membrane paradigm and the non-relativistic screen formalism, where it is simply $\gamma_u$. The pressure can thus vanish for non-trivial screen geometries leading to complex thermodynamic behaviour. The fluid expansion $\theta$ is directly related to the expansion in the time direction of the screen. The intuition is clear: if the screen expands with time, the 2d volume available to the fluid increases, and the fluid expands.

A new feature appearing in our work is the fluid twist or kinematic vorticity $\epsilon_{ab}$ which is equal to the twist $\omega_{ab}$ of the screen observers. The fluid twist measures rigid rotations of fluid lines with respect to the local inertial rest frame. Traditionally, quasi-local approaches to studying gravitational thermodynamics use shell observers whose 4-velocities are hypersurface orthogonal [5], and therefore have vanishing twist. In hydrodynamics however, $\epsilon_{ab}$ plays an important role in establishing the Kelvin-Helmholtz and Bernoulli theorems, as well as modelling turbulence, and is therefore a critical ingredient in attempting to understand the full non-linear dynamics of gravity in hydrodynamic terms.

With this dictionary in place, well-known results from relativistic hydrodynamics as well as the laws of thermodynamics can be used to study gravitational phenomena.

## 5. Thermodynamics of screen fluids

The thermodynamic system governed by (6) has more degrees of freedom than constraints, requiring additional information to complete. In hydrodynamics, this comes in the form of an equation of state $e(p)$ which characterizes the fluid. From the gravity point of view, $e \sim \theta_u$ and $p \sim \gamma_s$, so it is the screen evolution/geometry which fixes the equation of state of the fluid. One natural question that arises is "What screen evolutions give rise to physical equations of state?". As an example, one can construct a static, spherically symmetric screen in Minkowski space, and see that the equation of state becomes $e(p) = -2p$, which has the form $p(e) = \omega e$ with $\omega = -1/2$. Such equations of state appear in cosmology as models of dark energy fluids, which satisfy the strong energy condition but do not support classical perturbations. This fluid requires a negative chemical potential in order to have a positive entropy and temperature, suggesting that the underlying microscopic constituents are bosonic in nature. One can also examine what happens in a Schwarzschild background, where we find that the fluid retains its barotropic nature, but now supports classical perturbations (the speed of sound is positive) and has an entropy which increases monotonically with the screen radius, allowing one to map the fluid entropy to the volume of the curved spacetime enclosed by the screen.

These simple examples of screens already demonstrate some of the salient features of the dictionary, and illustrate the subtleties involved in constructing screens which have interpretations in terms of physical fluids. One task of this work is to extend the formalism to dynamic backgrounds, which generically lead to time-dependent equations of state, and better understand how phenomena such as gravitational waves manifest in the fluid. This will provide new insights into the thermodynamic nature of gravity from a quasi-local perspective, and also shed light on dissipative phenomena in relativistic hydrodynamics, where exact solutions are unavailable.

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