Bouncing and Emergent Cosmologies from Hamiltonian Analysis of Asymptotically Safe Quantum Gravity

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Recent results based on renormalization group approaches to Quantum Gravity suggest that the Newton's and Cosmological constants should be treated as a dynamical variables whose evolution depend on the characteristic energy scale of the system. An open question is how to embed this modified Einstein's theory in the Dirac's theory of constrained systems. In this talk, the Hamiltonian formalism for a renormalization-group scale dependent Newton's and Cosmological constants is discussed paying particular attention to Dirac's constraint analysis. It is shown that the algebra of the Dirac's constraints is closed under certain conditions. Brans-Dicke theory is also studied as a Dirac's Constrained Dynamical System and it is confronted and contrasted with modified Einstein Theory of General Relativity via Asymptotically Safe Quantum Gravity. Applications to the physics of the Early Universe is explicitly discussed assuming the framework of Asymptotic Safety. In particular, it is shown that in the Minisuperspace case with FLRW metric, RG improved Friedmann equations exibit Bouncing and Emergent Universes solutions. While, in the classical case, Emergent universe solutions hold for closed topologies (K=+1), in the sub-Planckian regime they hold also for flat (K=0) and open (K=-1) topologies.

I. INTRODUCTION

It is well known that classical General Relativity is a quite successful phenomenological theory at laboratory, solar system, galactic and extragalactic scales and in general for length scales $l \gg l_{Pl} \approx 10^{-33} cm$, where l_{Pl} is the Planck length. Singularity problems of Einstein's equations at Planck length and the quantum behaviour of matter and energy at small distances (high energy) suggest that a quantum version of the gravitational field (Quantum Gravity) should be found. There are many different approaches to Quantum Gravity: String Theory, Loop Quantum Gravity, Non-Commutative Geometry, Causal Dynamical Triangulations, Poset Theory, Asymptotic Safety etc.

As the Newton's constant has a negative mass dimension, the perturbative quantization of General Relativity leads to a (perturbative) non-renormalizable theory. In general, perturbative non-renormalizable theories have a number of counter terms which increase as the loop orders. This implies that the renormalization process introduces infinitely many parameters so that the resulting theory does not have any predictive power [1]. This is not a dead end, because a perturbatively non-renormalizable theory might be renormalizable under a generalized notion of renormalizability based on non-perturbative arguments. This non-perturbative renormalizability, introduced by K. Wilson [2], is related to the existence of a Non-Gaussian Fixed Point (NGFP) which guarantee the finiteness of the theory in the ultraviolet limit[3].

The *Asymptotics Safety* conjecture dates back to Weinberg [4]. He suggested that General Relativity might be a non-perturbatively renormalizable Quantum Field Theory if the gravitational RG-flow approaches a non-trivial fixed point in the high energy limit. He himself proved that NGFP exists in $2+\epsilon$ dimensions [4]. In d=4 a NGFP exists in the case of Einstein-Hilbert truncation [5]. The main idea of this approach is that if one has a classical action of Gravity, in the Riemannian case, coupled with a_i constants coupled to $O^i(x,g)$ operators, x and g being, respectively, the space-time coordinates and the metric tensor g [6],

$$S(M,g) = \int_M d^4x \sqrt{g} \sum_{i=0}^{\infty} a_i O^i(x,g) \quad , \tag{1}$$

M is the four dimensional differentiable manifold. The renormalizable group is defined once one fixes an infrared cutoff k and writes the renormalization group equations in terms of the dimensionless coupling constants $\tilde{a}_i(k)$ and

the β-functions in the following manner [6]

$$k\partial_k \tilde{a}_i(k) = \beta_i(\tilde{a}_1(k), \tilde{a}_2(k), \tilde{a}_3(k)...) \tag{2}$$

A point $\tilde{a}_\star$ is a NGFP if it is a non trivial zero of the beta-functions, that is $\beta_i(\tilde{a}_\star) = 0 \; \forall i$ and $\tilde{a}_\star \neq 0$.
Once one has found a the NGFP, the next step is to linearize previous equation [6]

$$k\partial_k \tilde{a}_i(k) = \sum_j B_{ij} \left(\tilde{a}_j(k) - \tilde{a}_{\star j}(k) \right) \;\;, \tag{3}$$

where one has assumed the following definitions:

$$B_{ij} \equiv \partial_j \beta_i(\tilde{a}_\star) \;\;, \;\;\; B = (B_{ij}) \;\;\;. \tag{4}$$

The general solution to the previous linear equation can be written in the following form

$$\tilde{a}_i(k) = a_{\star i} + \sum_I C_I V_i^I \left(\frac{k_0}{k} \right)^{\Theta_I} \;\;\;, \tag{5}$$

where V^I are right-eigenvectors, solutions of the eigenvalue equation (matrix equation)

$$B \; V^I = -\Theta_I V^I \tag{6}$$

Θ_I being the critical exponents. Now, notice that the fact that one assumes $\tilde{a}_i(k) \longmapsto a_{\star i}$ when $k \mapsto \infty$ requires that $C_I = 0 \forall I$ in case Re $\Theta_I < 0$. The Ultra-Violet(UV) critical surface S_{UV} is defined as the number of independent renormalization group trajectories hitting the fixed point as $k \mapsto \infty$. The dimension Δ_{UV} of this surface is the dimension of S_{UV}. Said in another way, the dimension of the critical surface is the number of independent attractive directions or, equivalently, the number of eigenvalues Θ with Re $\Theta_I > 0$. The resulting quantum theory has Δ_{UV} free parameters. If this number is finite, then the theory is predictive as a pertinent renormalizable model with Δ_{UV} renormalizable couplings.

These considerations hold, in general, but has been introduced for the perturbative renormalization group (RG). In the non perturbative case one starts from a Wilson-type, coarse-grained, free energy functional

$$\Gamma_k [g_{\mu\nu}] \;\;\;, \tag{7}$$

where k is the infrared cut-off. Γ_k contains all the quantum fluctuations with momenta $p > k$ and not yet of those with $p < k$. The modes $p < k$ are suppressed in the path-integral by a mass-square type term $R_k(p^2)$.

The behavior of the free-energy functional interpolates between $\Gamma_{k \mapsto \infty} = S$, S being the classical (bare) action, and $\Gamma_{k \mapsto 0} = \Gamma$, Γ being the standard effective action. Γ_k satisfies the RG-equation, called also the Wetterich equation [7],

$$k\partial_k \Gamma_k = \frac{1}{2} Tr \left[(\delta^2 \Gamma_k + R_k)^{-1} k\partial_k R_k \right] \tag{8}$$

In general, since this RG-equation is quite complicate, one adopts a powerlul non perturbative approximation scheme: truncate the space of the action functional and project the RG flow onto a finite dimensional space. That is to say, one consider that the free energy functional Γ_k, formally, can be expanded in the following way

$$\Gamma_k[\cdot] = \sum_{i=0}^N g_i(k) k^{d_i} I_i[\cdot] \;\;\;, \tag{9}$$

where $I_i[\cdot]$ are given "local or non local functionals" of the fields and $g_i(k)$. In the case of gravity, the following truncation ansatz is usually made:

$$I_0[g] = \int d^4x \sqrt{g} \;\;\;, I_1[g] = \int d^4x \sqrt{g} R \;\;\;, I_2[g] = \int d^4x \sqrt{g} R^2 \;\;\;, \text{etc.} \tag{10}$$

The simplest truncation is the Einstein-Hilbert truncation which looks like

$$\Gamma_k = -\frac{1}{16\pi G_k} \int d^4x \left(R - 2\bar{\lambda}_k \right) + \text{g.f.} + \text{g.t.} \quad , \tag{11}$$

here g.f. means classical gauge fixing terms, while g.h. are ghost terms. There are two running parameters G_k, the Newton constant, which can be written in a dimensionless way as $g(k) = k^2 G_k$. In the same manner, the cosmological constant $\bar{\lambda}_k$ becomes $\lambda(k) = \bar{\lambda}_k / k^2$.

Inserting this ansatz into the flow (Wetterich) equation, one obtains "a projection" onto e finite dimensional space [3]

$$Tr[...] = (...) \int \sqrt{g} + (...) \int \sqrt{g} R + ... \quad , \tag{12}$$

and then the following finite-dimensional RG equations

$$k\partial_k g(k) = \beta_g(g, \lambda)$$
$$k\partial_k \lambda(k) = \beta_\lambda(g, \lambda) . \tag{13}$$

The solutions of this equations provide the scaling relation for the a-dimensional gravitational constant $g(k)$ and the a-dimensional cosmological constant $\lambda(k)$.

In the same direction ADM formalism for Black Holes could result quite enlightening . Here a completely different symmetry implies a different ADM foliation, which could, eventually, help to answer previous questions.

II. FUTURE DIRECTIONS

Hamiltonian (ADM) analysis of RG improved Einstein-Hilbert action with G and Λ as external, non geometrical field, can be performed. It can be shown that if one requires that this theory behaves like the Hamiltonian theory of Einstein General Relativity, that is the momentum constraints and the Hamiltonian constraint be the generators respectively of the space diffeomorphisms on Σ and the time diffeomorphisms, then one cannot start from the ADM-metric but from ADM metric in Gaussian normal coordinates.

An immediate application of the above considerations is FLRW cosmology in the minisupersapce approach using Dirac's constraint analysis. It generates sub-Planckian cosmological models via Asymptotic Safety. They exhibit bouncing and emergent Universes. The latter ones are solution of the equations of motion also in cases $K = -1, 0$, that are impossible to draw from classical General Relativity.

Although this analysis shows that RG improved Einstein-Hilbert action with G and Λ as external fields can be cast in the Hamiltonian formalism only in the case of ADM metric in Gaussian normal coordinates, one can still legitimately ask if there exists cases and/or particular foliations in which one does not need to loose space diffeomorphisms in order to make sense of the Hamiltonian formalism. In order to throw light on this issue, it could be useful, following the suggestions of section 2, to study the Hamiltonian formalism of the Branse-Dicke theory.

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