Extended mimetic gravity: Hamiltonian analysis and gradient instabilities

Kazufumi Takahashi* and Tsutomu Kobayashi

Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan
**E-mail: takahashik@rikkyo.ac.jp*

We propose a novel class of degenerate higher-order scalar-tensor theories as an extension of mimetic gravity. By performing a noninvertible conformal transformation on "seed" scalar-tensor theories which may be nondegenerate, we can generate a large class of theories with at most three physical degrees of freedom. We identify a general seed theory for which this is possible. Cosmological perturbations in these extended mimetic theories are also studied. It is shown that either of tensor or scalar perturbations is generically plagued with ghost/gradient instabilities. **See Ref. 1 for more details.**

Keywords: Scalar-Tensor Theories; Mimetic Gravity; Cosmological Perturbations.

1. Introduction

When one constructs a field theory with higher derivatives, a guiding principle comes from the theorem of Ostrogradsky [2], which states that any theory described by a nondegenerate higher derivative Lagrangian has unstable extra degrees of freedom (DOFs), i.e., Ostrogradsky ghost. Therefore, a theory without Ostrogradsky ghost, often referred to as a *healthy* theory, must have a degenerate Lagrangian.

Within (single-field) scalar-tensor theories in four dimensions, the Horndeski theory [3] (or its equivalent formulation known as generalized Galileons [4,5]) provides a basic ground for studying a wide class of such healthy theories having three DOFs, since it is the most general scalar-tensor theory that yields second-order Euler-Lagrange equations. There are further possibilities of healthy theories beyond the Horndeski class, such as Gleyzes-Langlois-Piazza-Vernizzi (GLPV) theories [6] and quadratic/cubic degenerate higher-order scalar-tensor (DHOST) theories [7–9]. Those quadratic/cubic DHOST theories form the broadest class of healthy scalar-tensor theories known so far. However, these theories are obtained under the assumption that the Lagrangian depends on up to quadratic/cubic order in $\nabla_\mu\nabla_\nu\phi$ (hence the name "quadratic/cubic DHOST"), and thus the very boundary of healthy scalar-tensor theories remains unclear.

In light of this situation, we explore the possibility to construct a new class of DHOST theories by use of conformal/disformal transformation of the metric, which has been employed for investigating the relation between the known scalar-tensor models. To this end, we perform a *noninvertible* conformal transformation on generic scalar-tensor theories that could possess an unwanted extra DOF. Here, the noninvertibility of the transformation is crucial because otherwise the resultant theory could also have an extra DOF [10]. Since the formulation of the new theory can be regarded as an extension of the mimetic gravity model introduced in Ref. 11, we call the theory *extended mimetic gravity*. We show that these models have at most three DOFs based on Hamiltonian analysis and thus form a broad class of DHOST theories, most of which lie outside the quadratic/cubic DHOST class. We also study the linear stability of cosmological perturbations in our extended mimetic gravity and show that the models obtained in the above explained way generically exhibit the problem of ghost/gradient instabilities.

2. Extended Mimetic Gravity

We start from the following general scalar-tensor theory:

$$S_{\rm seed} = \int d^4x\sqrt{-g}\,[f_2(\phi,X)\mathcal{R} + f_3(\phi,X)\mathcal{G}^{\mu\nu}\phi_{\mu\nu} + F(g_{\mu\nu},\phi,\phi_\mu,\phi_{\mu\nu})]\,, \quad (1)$$

with $\mathcal{R}$ and $\mathcal{G}_{\mu\nu}$ being the four-dimensional Ricci scalar and Einstein tensor, $\phi_\mu \equiv \nabla_\mu\phi$, $\phi_{\mu\nu} \equiv \nabla_\mu\nabla_\nu\phi$, and $X \equiv \phi^\mu\phi_\mu$. Note that, for a generic choice of the functions f_2, f_3, and F, the theory yields Ostrogradsky ghost. Using the action of the form (1) as a seed, we perform the following conformal transformation:

$$g_{\mu\nu} \;\to\; \tilde{g}_{\mu\nu} = -Xg_{\mu\nu}, \quad (2)$$

Here, $\tilde{g}_{\mu\nu}$ is identified as the metric in the original frame (1), while $g_{\mu\nu}$ is now the metric of the new theory, which we call *extended mimetic gravity*. The transformation (2) is noninvertible as the right-hand side is invariant under conformal transformation of $g_{\mu\nu}$. As a result, the new theory acquires a local conformal symmetry.

To study whether the so-obtained extended mimetic gravity models possess a problematic extra DOF or not, one needs to perform a Hamiltonian analysis. In doing so, we write the action in the ADM language, which becomes of the form

$$S_{\rm EMG} = \int dtd^3x\left[N\sqrt{\gamma}L_{\rm EMG} + \Lambda(NA_* + N^iD_i\phi - \dot{\phi})\right], \quad (3)$$

where $L_{\rm EMG}$ is some function of $(\gamma_{ij}, R_{ij}, \phi, A_*; V_{ij}; D_i)$. Here, we have defined

$$V_{ij} \equiv K_{ij} + \frac{\dot{A}_* - D^i\phi D_iN - N^iD_iA_*}{NA_*}\gamma_{ij}, \quad (4)$$

and introduced an auxiliary variable A_* with a Lagrange multiplier Λ so that second-order time derivatives do not appear explicitly in the action. Reflecting the aforementioned local conformal symmetry, $\dot{\gamma}_{ij}$ and $\dot{A}_*$ (namely, $\ddot{\phi}$) appear only in a special combination V_{ij}. When one proceeds to a Hamiltonian analysis, this relation leads to an additional primary constraint, which turns out to be first class. Thus, it kills the otherwise existing unwanted DOF, leaving only three DOFs. For a more rigorous and detailed analysis, see Ref. 1.

3. Cosmological Perturbations

To see whether the general mimetic theories are phenomenologically viable or not, let us analyze perturbations around the Friedmann-Lemaître-Robertson-Walker (FLRW) background in the mimetic gravity models.

For simplicity, we take the unitary gauge to write $\phi = t$ and impose the constraint $X = -1$ which fixes the conformal gauge DOF. As a consequence, any function of (ϕ, X) can be regarded as a function of t only. We also have $N = 1$ since X is written in terms of N as $X = -1/N^2$ in the unitary gauge. Under this setup, the action can be recast in the following form:

$$S_{\rm EMG} = \int dtd^3x\sqrt{\gamma}\left[\left(f_2 - \frac{1}{2}\dot{f}_3\right)R + \mathcal{F}(t, \mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3, \cdots, \mathcal{K}_\ell)\right], \tag{5}$$

where $\mathcal{F}$ is some function of t and $\mathcal{K}_n \equiv K^{i_1}_{i_2}K^{i_2}_{i_3}\cdots K^{i_n}_{i_1}$ $(n = 1, 2, \cdots, \ell)$. It is useful to define the first and second derivatives of $\mathcal{F}$ respectively as

$$\mathcal{F}_n \equiv \frac{\partial \mathcal{F}}{\partial \mathcal{K}_n}, \quad \mathcal{F}_{mn} \equiv \frac{\partial^2 \mathcal{F}}{\partial \mathcal{K}_m \partial \mathcal{K}_n}. \tag{6}$$

Now we substitute the following metric ansatz to the action (5),

$$N = 1, \quad N_i = \partial_i \chi, \quad \gamma_{ij} = a^2 e^{2\zeta}\left(\delta_{ij} + h_{ij} + \frac{1}{2}h_{ik}h_{jk} + \cdots\right), \tag{7}$$

where χ and ζ are scalar perturbations and h_{ij} denotes a transverse-traceless tensor perturbation. The background EOM is given by

$$\dot{\mathcal{P}} + 3H\mathcal{P} - \mathcal{F} = 0, \quad \mathcal{P} \equiv \sum_{n=1}^{\ell} n\, H^{n-1}\mathcal{F}_n, \tag{8}$$

where $H \equiv \dot{a}/a$ is the Hubble parameter and $\mathcal{F}_n$ are evaluated at the background, $\mathcal{K}_n = 3H^n$. This equation will be used to simplify the expressions of the quadratic actions for the tensor and scalar perturbations below.

The quadratic action for the tensor perturbation h_{ij} is given by

$$S^{(2)}_{\rm T} = \int dtd^3x \frac{a^3}{4}\left[\mathcal{B}\dot{h}^2_{ij} - \mathcal{E}\frac{(\partial_k h_{ij})^2}{a^2}\right], \tag{9}$$

where

$$\mathcal{B} \equiv \sum_{n=2}^{\ell} \frac{n(n-1)}{2} H^{n-2}\mathcal{F}_n, \quad \mathcal{E} = f_2 - \frac{1}{2}\dot{f}_3. \tag{10}$$

The tensor perturbations are stable if $\mathcal{B} > 0$ and $\mathcal{E} > 0$.

The quadratic action for the scalar perturbations ζ and χ is

$$S^{(2)}_{\rm S} = \int dtd^3x a^3\left[\frac{3}{2}(3\mathcal{A} + 2\mathcal{B})\dot{\zeta}^2 + 2\mathcal{E}\frac{(\partial_k\zeta)^2}{a^2} + \frac{1}{2}(\mathcal{A} + 2\mathcal{B})\left(\frac{\partial^2\chi}{a^2}\right)^2 - (3\mathcal{A} + 2\mathcal{B})\dot{\zeta}\frac{\partial^2\chi}{a^2}\right], \tag{11}$$

where

$$\mathcal{A} \equiv \sum_{m=1}^{\ell}\sum_{n=1}^{\ell} m\, n\, H^{m+n-2}\mathcal{F}_{mn}. \tag{12}$$

Provided that $\mathcal{A}+2\mathcal{B}\neq 0$, χ can be eliminated from the action by use of its EOM. Then, we are left with the following quadratic action for curvature perturbation:

$$S_{\rm S}^{(2)}=2\int dtd^3x\,a^3\left[\frac{\mathcal{B}(3\mathcal{A}+2\mathcal{B})}{\mathcal{A}+2\mathcal{B}}\dot{\zeta}^2+\mathcal{E}\frac{(\partial_k\zeta)^2}{a^2}\right]. \tag{13}$$

Written in this form, one notices that the stability condition for the tensor perturbations, $\mathcal{E}>0$, is not compatible with the stability for the scalar perturbation, $\mathcal{E}<0$. This indicates that either of the tensor or scalar perturbations exhibits gradient instabilities, even if one circumvents ghosts by choosing the coefficients in front of the time derivative terms in Eqs. (9) and (13) to be positive.

One would notice that if $\mathcal{B}(\mathcal{A}+2\mathcal{B})(3\mathcal{A}+2\mathcal{B})=0$ then the scalar perturbations appear to be nondynamical. This is indeed the case if we choose a Horndeski (or GLPV) theory as seed, where $\mathcal{A}+2\mathcal{B}=0$. In this case, it is important to take into account the presence of matter fields other than ϕ to discuss the viability of mimetic cosmology. If another scalar field ψ is added to the seed Lagrangian, one can show that the scalar perturbations revive, i.e., there are now two scalar DOFs, and one of them is a ghost.

Thus, we have established that all the mimetic gravity models with three DOFs obtained so far are plagued with ghost/gradient instabilities on a cosmological background (except for the special case of strongly-coupled scalar perturbations). More detailed discussion is found in Ref. 1 (see also Ref. 12).

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