New 2D dilaton gravity for nonsingular black holes

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We construct a two-dimensional action that is an extension of spherically symmetric Lovelock gravity. In spite that the action contains arbitrary functions of the areal radius and the norm squared of its gradient, the field equations are second order and obey the Birkhoff's theorem. Similar to the spherically symmetric Lovelock gravity, the field equations admit the generalized Misner-Sharp mass that determines the form of the vacuum solution. The arbitrary functions in the action allow for vacuum solutions that describe a larger class of nonsingular black-hole spacetimes than previously available.

Keywords: Nonsingular black hole; Two-dimensional effective theory.

1. 2D effective actions for spherically symmetric spacetimes

The metric for $n(\geq 3)$-dimensional spherically symmetric spacetimes is given by

$$\begin{aligned} \mathrm{d}s^2_{(n)} =& g_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu \\ =& \bar{g}_{AB}(y)\mathrm{d}y^A\mathrm{d}y^B + R(y)^2\mathrm{d}\Omega^2_{(n-2)}, \end{aligned} \tag{1}$$

where $\bar{g}_{AB}(y)$ $(A, B = 0, 1)$ is the general two-dimensional (2D) Lorentzian metric, $\mathrm{d}\Omega^2_{(n-2)}$ is the line-element on the unit $(n-2)$-sphere, and R is the areal radius.

After imposing spherical symmetry and integrating out the angular variables, the general $n(\geq 3)$-dimensional gravitational action,

$$I_n = \frac{1}{16\pi G_{(n)}} \int \mathrm{d}^n x\sqrt{-g}\mathcal{L}(\mathcal{R}, \mathcal{R}_{\mu\nu}, \mathcal{R}_{\mu\nu\rho\sigma}), \tag{2}$$

reduces to a 2D effective action. The variation of this effective 2D action will give the same equations of motion for the original action (2) [1–3].

We adopt units such that $c = \hbar = 1$ and $G_{(n)}$ denotes the n-dimensional gravitational constant. In the following, D_A and $\mathcal{R}[\bar{g}]$ denote the covariant derivative and the Ricci scalar with respect to $\bar{g}_{AB}$, respectively. We also define $(DR)^2 := (D_A R)(D^A R)$ and a length parameter l proportional to the Planck length as $l^{n-2} := 16\pi G_{(n)}/\mathcal{A}_{(n-2)}$, where $\mathcal{A}_{(n-2)}$ is the volume of a unit $(n-2)$-sphere. The complete analysis for the results presented here is available in Ref. 4.

1.1. *Effective 2D action for Einstein gravity and its generalization*

The Einstein-Hilbert action for general relativity corresponds to $\mathcal{L} = \mathcal{R}$ in the action (2) and its effective 2D action takes the form

$$\begin{aligned} I_{\rm EH(2)} =& \frac{1}{l^{n-2}} \int {\rm d}^2 y \sqrt{-\bar{g}} \Big\{ R^{n-2} \mathcal{R}[\bar{g}] \\ & + (n-2)(n-3) R^{n-4} (DR)^2 + (n-2)(n-3) R^{n-4} \Big\}. \end{aligned} \tag{3}$$

By the Birkhoff's theorem, the unique vacuum solution with spherical symmetry is the well-known Schwarzschild-Tangherlini solution.

A natural way to generalize the spherically symmetric action (3) in Einstein gravity is the following 2D dilaton gravity:

$$I_{(2)} = \frac{1}{l^{n-2}} \int {\rm d}^2 y \sqrt{-\bar{g}} \Big\{ \phi(R) \mathcal{R}[\bar{g}] + h(R)(DR)^2 + V(R) \Big\}, \tag{4}$$

where $\phi(R)$, $h(R)$, and $V(R)$ are arbitrary functions of a scalar field R. (See Ref. 5 for a review on this class of 2D gravity.) This standard 2D dilaton gravity theory (4) obeys the Birkhoff's theorem [6]. Namely, the general vacuum solution has the following form:

$${\rm d}s^2 = -f(R){\rm d}t^2 + f(R)^{-1}{\rm d}R^2. \tag{5}$$

If one chooses $h(R) = V(R) = \phi_{,RR}(R)$, the metric function is given by

$$f(R) = 1 - \frac{l^{n-2} M}{j(R)}, \quad \left(j(R) := \int V(R) {\rm d}R \right). \tag{6}$$

Especially, in the case with $n = 4$ and $\phi_{,R} = j(R) = (R^2 + l^2)^{3/2}/R^2$, the general vacuum solution is the well-known Bardeen metric [7]:

$${\rm d}s^2_{(4)} = -\left(1 - \frac{l^2 M R^2}{(R^2+l^2)^{3/2}}\right) {\rm d}t^2 + \left(1 - \frac{l^2 M R^2}{(R^2+l^2)^{3/2}}\right)^{-1} {\rm d}R^2 + R^2 {\rm d}\Omega^2_{(2)}. \tag{7}$$

The Bardeen spacetime (7) is certainly nonsingular everywhere, however, this class of nonsingular black holes are considered to be unphysical. This is because the metric (7) violates the *limiting curvature conjecture*, which asserts that the curvature invariants are bounded by some fundamental value in a viable fundamental theory [8]. In fact, to the best of our knowledge, this limiting curvature condition cannot be fulfilled within the framework of the action for pure 2D dilaton gravity (4). This is the main reason why we consider a more general class of 2D dilaton gravity.

1.2. *Effective 2D action for Lovelock gravity and its generalization*

Lovelock gravity is a natural generalization of general relativity in arbitrary dimensions as a second-order quasilinear theory of gravity [9]. The second-order field equations ensure the ghost-free nature of the theory and Lovelock gravity reduces to general relativity with a cosmological constant in four dimensions.

In the action (2), Lovelock gravity in vacuum corresponds to

$$\mathcal{L} = \sum_{p=0}^{[n/2]} 2^{-p}\alpha_{(p)}\delta^{\mu_1\cdots\mu_p\nu_1\cdots\nu_p}_{\rho_1\cdots\rho_p\sigma_1\cdots\sigma_p}\mathcal{R}_{\mu_1\nu_1}{}^{\rho_1\sigma_1}\cdots\mathcal{R}_{\mu_p\nu_p}{}^{\rho_p\sigma_p}, \tag{8}$$

where $\delta^{\mu_1\cdots\mu_p}_{\rho_1\cdots\rho_p} := p!\delta^{\mu_1}_{[\rho_1}\cdots\delta^{\mu_p}_{\rho_p]}$, and its effective 2D action was obtained[10,11] as

$$\begin{aligned} I_{\mathrm{L}(2)} =& \frac{1}{l^{n-2}}\int \mathrm{d}^2y\sqrt{-\bar{g}}R^{n-2}\sum_{p=0}^{[n/2]}\frac{(n-2)!}{(n-2p)!}\alpha_{(p)} \\ &\times\Biggl[p\mathcal{R}[\bar{g}]R^{2-2p} + (n-2p)(n-2p-1)\Bigl\{(1-Z)^p + 2pZ\Bigr\}R^{-2p} \\ &+ p(n-2p)R^{1-2p}\Bigl\{1-(1-Z)^{p-1}\Bigr\}(D_AR)\frac{(D^AZ)}{Z}\Biggr], \end{aligned} \tag{9}$$

where we have defined $Z := (DR)^2$. The Birkhoff's theorem in Lovelock gravity shows that, under several technical assumptions, the unique vacuum solution is given by the Schwarzschild-Tangherlini-type solution[12,13].

In analogy with the action (4), we now propose[4] the following natural extension of the spherically symmetric Lovelock action (9)

$$I_{\mathrm{XL}} = \frac{1}{l^{n-2}}\int \mathrm{d}^2y\sqrt{-\bar{g}}\Bigl\{\phi(R)\mathcal{R}[\bar{g}] + \eta(R,Z) + \chi(R,Z)(D_AR)\frac{(D^AZ)}{Z}\Bigr\}, \tag{10}$$

where $\eta(R,Z)$ and $\chi(R,Z)$ are as yet arbitrary functions of a scalar field R and Z. For any given $\phi(R)$ and $\chi(R,Z)$, one can choose the function $\eta(R,Z)$ as

$$\phi_{,RR} = \eta_{,Z} - \chi_{,R}, \tag{11}$$

where a comma denotes the partial derivative, so that the field equations obey the Birkhoff's theorem for $Z = (DR)^2 \neq 0$ and $\chi - \phi_{,R} \neq 0$. Then the resulting general vacuum solution has the following form:

$$\mathrm{d}s^2 = -f(R)\mathrm{d}t^2 + f(R)^{-1}\mathrm{d}R^2. \tag{12}$$

Actually, the condition (11) ensures the existence of the generalized Misner-Sharp mass which satisfies the unified first law. Under the condition (11), the existence of Minkowski vacuum requires

$$\eta(R,1) = 2\phi_{,RR}. \tag{13}$$

2. Designing nonsingular black holes

Now we show how to construct specific nonsingular black holes as exact solutions by making appropriate choices for the functions in the action (10). We are interested in constructing nonsingular black holes that satisfy the limiting curvature condition, namely curvature invariants are everywhere bounded for arbitrarily large M.

Such an example is the following Hayward nonsingular black hole[14]:

$$f(R) = 1 - \frac{l^2 M R^2}{R^3 + l^4 M}, \tag{14}$$

of which generalization in n dimensions is given by[4]

$$f(R) = 1 - \frac{l^{n-2} M R^2}{R^{n-1} + l^n M}. \tag{15}$$

This n-dimensional Hayward black hole (15) is the unique vacuum solution in the theory with

$$\eta(R,Z) = 2\phi_{,RR}Z + \frac{(n-3)l^{n-2}R^n(1-Z) - (n-1)l^n R^{n-2}(1-Z)^2}{\{l^{n-2}R^2 - l^n(1-Z)\}^2},$$
$$\chi(R,Z) = \phi_{,R} - \frac{l^{n-2}R^{n+1}}{\{l^{n-2}R^2 - l^n(1-Z)\}^2}. \tag{16}$$

Also, the following Bardeen-type nonsingular black hole[4]

$$f(R) = 1 - \frac{l^{n-2} M R^2}{(R^2 + M^{2/(n-1)} l^{2n/(n-1)})^{(n-1)/2}} \tag{17}$$

or new nonsingular black hole[4]

$$f(R) = 1 + \frac{R^{n+1}}{2l^{n+2}M}\left(1 - \sqrt{1 + \frac{4l^{2n}M^2}{R^{2(n-1)}}}\right) \tag{18}$$

can be the unique vacuum solution in the theory with a suitable choice of $\eta(R,Z)$ and $\chi(R,Z)$.

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