Gravitational waves in the most general teleparallel theories

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In my talk I will consider gravitational waves in the most general class of teleparallel theories, where gravity is described via torsion, and symmetric teleparallel theories, where gravity is attributed to non-metricity. Both classes depend on a number of constant parameters. The gravitational wave will be treated as a linear perturbation around a flat background. I will discuss the possible polarizations of gravitational waves which are allowed by the linearized field equations.

Keywords: Newman-Penrose formalism; Teleparallel theories; Gravitational waves.

1. Introduction

We use the following notation. Latin letters $a, b, ...$ are Lorentz indices, greek letters $\mu, \nu, ...$ are spacetime coordinate indices. The Minkowski metric has components $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$.

The fundamental variables in theories of gravity formulated in terms of teleparallelism are the tetrad 1-forms θ^a, their dual vector fields e_a and the curvature free spin connection $\omega^a{}_b$ generated by local Lorentz transformations $\Lambda^a{}_b$.

The building block of Lagrange densities is the torsion of the spin-connection given by

$$T^a = \mathrm{D}\theta^a = (\partial_\mu \theta^a{}_\nu + \omega^a{}_{b\mu}\theta^b{}_\nu)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \,, \tag{1}$$

where the spin covariant derivative D ensures a covariant transformation behaviour under local Lorentz transformations of the tetrad. In the following we will use the torsion components with spacetime indices only obtained via $T^\alpha{}_{\mu\nu} = T^a{}_{\mu\nu} e_a{}^\alpha$.

2. NGR Lagrange density and field equations

The class of gravity theories called *New General Relativity* (NGR) [1,2] is defined by the most general Lagrange densities which are quadratic in the torsion. They can be displayed in a closed form by introducing three real parameters c_1, c_2 and c_3 parametrizing the different NGR theories

$$L(\theta, \partial\theta, \lambda, \partial\lambda) = |\theta| \left(c_1 T^\rho{}_{\mu\nu} T_\rho{}^{\mu\nu} + c_2 T^\rho{}_{\mu\nu} T^{\nu\mu}{}_\rho + c_3 T^\rho{}_{\mu\rho} T^{\sigma\mu}{}_\sigma \right) . \tag{2}$$

To analyse the propagation of gravitational waves for NGR gravity we derive the linearized field equations of the theory. To do so we fix Cartesian coordinates $(x^\mu, \mu = 0, \ldots, 3)$ and make the following perturbative Ansatz for the tetrad and the Lorentz transformation defining the spin connection

$$\theta^a{}_\mu = \delta^a_\mu + \varepsilon\ \mathrm{u}^a{}_\mu, \quad e_a{}^\mu = \delta^\mu_a + \varepsilon\ \mathrm{v}_a{}^\mu, \quad \Lambda^a{}_b = \delta^a_b + \varepsilon\ \mathrm{w}^a{}_b\,, \tag{3}$$

where ε is a perturbation parameter.

To proceed we introduce the new variable $\mathrm{x}_{\beta\sigma} = \mathrm{u}_{\beta\sigma} - \mathrm{w}_{\beta\sigma}$ and decompose it into its symmetric and antisymmetric part $\mathrm{x}_{\beta\sigma} = s_{\beta\sigma} + a_{\beta\sigma}$ which allows us to analyse the field equations further. Using this, the linearized field equations take the following form

$$\begin{aligned} 0 = E^{\tau\kappa} &= \partial_\rho\big[(2c_1 - c_2)\partial^\rho a^{\tau\kappa} - (2c_1 - c_2)\partial^\kappa a^{\tau\rho} + (2c_2 + c_3)\partial^\tau a^{\rho\kappa}\big] \\ &+ \partial_\rho\big[(2c_1 + c_2)\partial^\rho s^{\tau\kappa} - (2c_1 + c_2)\partial^\kappa s^{\tau\rho} + c_3\big(\eta^{\tau\kappa}(\partial^\rho s^\beta{}_\beta - \partial_\lambda s^{\rho\lambda}) - \eta^{\tau\rho}(\partial^\kappa s^\beta{}_\beta - \partial^\tau s^{\rho\kappa})\big)\big]\,. \end{aligned} \tag{4}$$

We note that indices can be lowered and raised at the first order perturbations with Minkowski metric only. In the following we will deduce the polarization modes of the perturbations from these field equations.

3. Newman-Penrose formalism and polarizations

The main ingredient of the Newman-Penrose formalism [3] is the choice of a particular complex double null basis of the tangent space. In the following, we will use the notation of[4] and denote the basis vectors by $l^\mu, n^\mu, m^\mu, \bar{m}^\mu$. In terms of the canonical basis vectors of the Cartesian coordinate system they are defined as

$$l = \partial_0 + \partial_3\,, \quad n = \frac{1}{2}(\partial_0 - \partial_3)\,, \quad m = \frac{1}{\sqrt{2}}(\partial_1 + i\partial_2)\,, \quad \bar{m} = \frac{1}{\sqrt{2}}(\partial_1 - i\partial_2)\,. \tag{5}$$

We now consider a plane wave propagating in the positive x^3 direction, which corresponds to a single Fourier mode. The wave covector then takes the form $k_\mu = -\omega l_\mu$ and the symmetric and antisymmetric parts of the tetrad perturbations can be written in the form

$$s_{\mu\nu} = S_{\mu\nu}e^{i\omega u}\,, \quad a_{\mu\nu} = A_{\mu\nu}e^{i\omega u}\,, \tag{6}$$

where we introduced the retarded time $u = x^0 - x^3$ and the wave amplitudes are denoted $S_{\mu\nu}$ and $A_{\mu\nu}$.

It follows from our choice of the matter coupling that test particles follow the geodesics of the metric, and hence the autoparallel curves of the Levi-Civita connection. The effect of a gravitational wave on an ensemble of test particles, or any other type of gravitational wave detector, therefore depends only on the Riemann tensor derived from the Levi-Civita connection. As shown in [5], the Riemann tensor of a plane wave is determined completely by the six so-called electric components.

For the wave (6), these can be written as

$$\Psi_2 = -\frac{1}{6}R_{nlnl} = \frac{1}{12}\ddot{s}_{ll}\,, \quad \Psi_3 = -\frac{1}{2}R_{nln\bar{m}} = -\frac{1}{2}\overline{R_{nlnm}} = \frac{1}{4}\ddot{s}_{l\bar{m}} = \frac{1}{4}\overline{\ddot{s}_{lm}}\,,$$
$$\Psi_4 = -R_{n\bar{m}n\bar{m}} = -\overline{R_{nmnm}} = \frac{1}{2}\ddot{s}_{\bar{m}\bar{m}} = \frac{1}{2}\overline{\ddot{s}_{mm}}\,, \quad \Phi_{22} = -R_{nmn\bar{m}} = \frac{1}{2}\ddot{s}_{m\bar{m}}\,, \tag{7}$$

where dots denote derivatives with respect to u. We now examine which of the components (7) may occur for gravitational waves satisfying the linearized field equations (4).

Inserting the wave ansatz (6) and writing the gravitational field strength tensor in the Newman-Penrose basis, we find that the eight component equations are satisfied identically. The remaining eight component equations give modes correspond to classes $\mathrm{N}_2, \mathrm{N}_3$, III_5, II_6 , depend on c_1, c_2, c_3.

4. Generalized Symmetric Teleprallel Theories of Gravity

We take the metric $g_{\mu\nu}$ and connection $\Gamma^{\alpha}{}_{\sigma\omega}$ as independent variables and consider the Lagrangian density in the symmetric teleparallelism [6,7]

$$\mathcal{L}_G = \frac{1}{2}\sqrt{-g}Q^{\alpha}{}_{\mu\nu}(c_1 Q_{\alpha}{}^{\mu\nu} + c_2 Q^{\mu}{}_{\alpha}{}^{\nu} + c_3 g^{\mu\nu}Q_{\alpha}$$
$$+ c_4\delta^{\mu}_{\alpha}\tilde{Q}^{\nu} + c_5\delta^{\mu}_{\alpha}Q^{\nu}) + \lambda_{\alpha}{}^{\beta\mu\nu}R^{\alpha}{}_{\beta\mu\nu} + \lambda_{\alpha}{}^{\mu\nu}T^{\alpha}{}_{\mu\nu}\,. \tag{8}$$

where so called non-metricity is given by

$$Q_{\alpha\mu\nu} \equiv \nabla_{\alpha}g_{\mu\nu}\,. \tag{9}$$

We work in the coincident gauge, where the connection coefficients are zero $\Gamma^{\alpha}{}_{\sigma\omega} = 0$, and consider a perturbation of the metric around the Minkowski metric

$$g_{\mu\nu} \;=\; \eta_{\mu\nu} + \mathrm{x}_{\mu\nu}\,. \tag{10}$$

The connection is metric compatible and torsion-free. The linearized field equations read

$$0 = 2c_1\eta^{\alpha\sigma}\partial_{\alpha}\partial_{\sigma}\mathrm{x}_{\mu\nu} + (c_2+c_4)\eta^{\alpha\sigma}(\partial_{\alpha}\partial_{\mu}\mathrm{x}_{\sigma\nu} + \partial_{\alpha}\partial_{\nu}\mathrm{x}_{\sigma\mu})$$
$$+2c_3\eta_{\mu\nu}\eta^{\tau\omega}\eta^{\alpha\sigma}\partial_{\alpha}\partial_{\sigma}\mathrm{x}_{\tau\omega} + c_5(\eta_{\mu\nu}\eta^{\omega\gamma}\eta^{\alpha\sigma}\partial_{\alpha}\partial_{\omega}\mathrm{x}_{\sigma\gamma} + \eta^{\omega\sigma}\partial_{\mu}\partial_{\nu}\mathrm{x}_{\omega\sigma})\,. \tag{11}$$

Alternatively, applying the same procedure we get $\mathrm{N}_2, \mathrm{N}_3$, III_5, II_6 classes, depend on c_2, c_4, c_5, while the terms come with c_1, c_3 vanish.

5. Conclusion

We have seen that depending on the constant parameters we obtain for both theory the E_2 class II_6, III_5, N_3 or N_2. We have also seen that there exists a family of theories besides TEGR which is of class N_2 and thus exhibits the same two tensor modes as in general relativity. Theories in this class therefore cannot be distinguished from general relativity by observing the polarizations of gravitational waves alone.

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