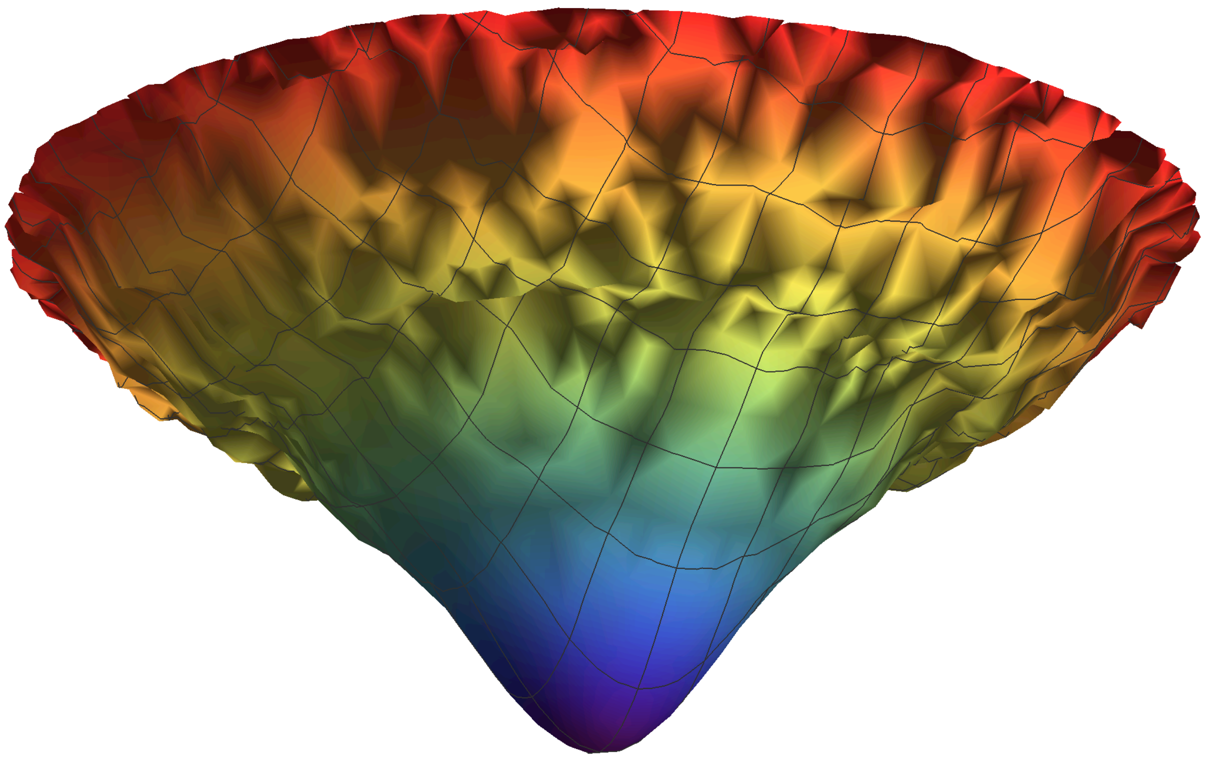
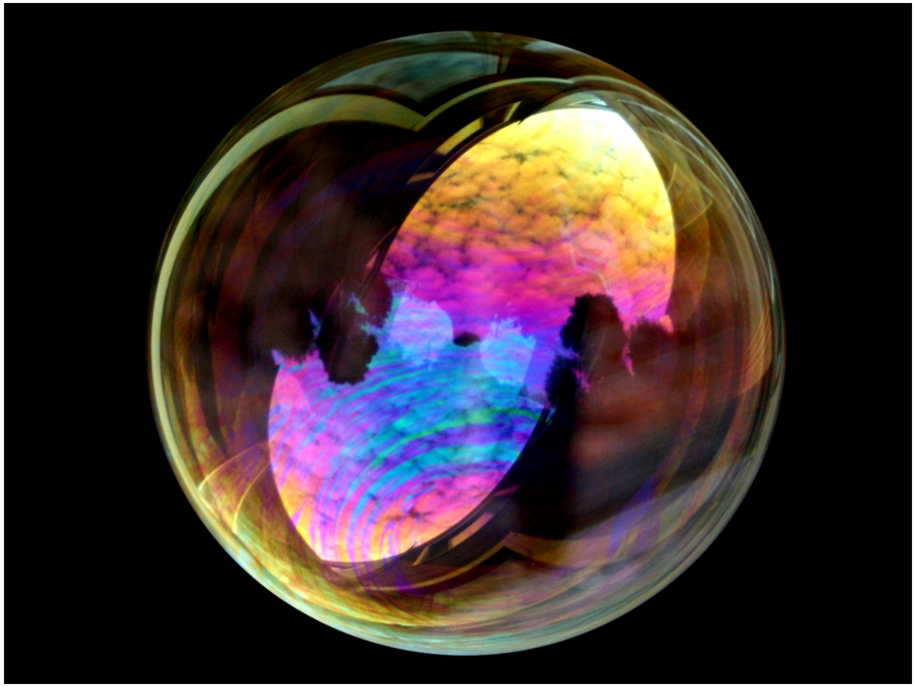
No Smooth Beginning for Spacetime



Jean-Luc Lehners
Max Planck Institute for Gravitational Physics
Albert Einstein Institute
With Job Feldbrugge and Neil Turok: 1703.02076 (PRD), 1705.00192 (PRL), 1709.03171 (PRD), 1805.01609
With Alice Di Tucci: 1806.07134



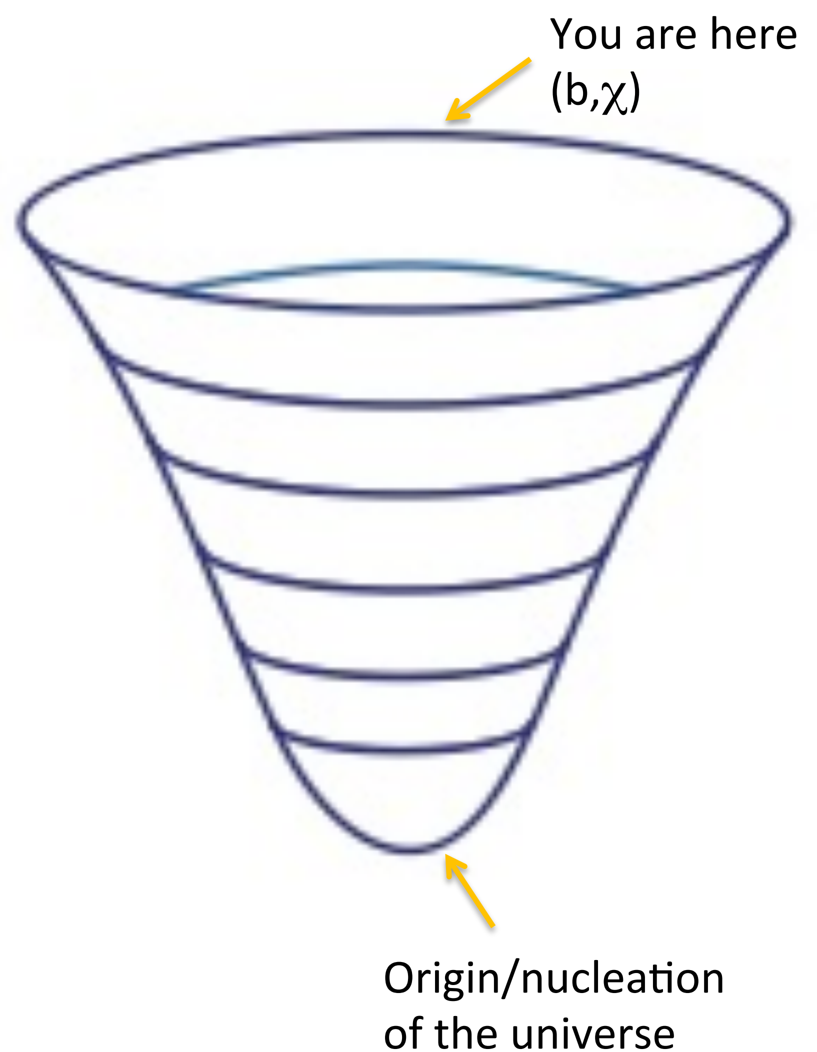
- There is an old and attractive idea that a closed universe can nucleate out of "nothing", since the total Hamiltonian vanishes, with all total charges zero

[Lemaître; Tryon; Brout, Englert & Gunzig]

No-boundary and tunneling proposals

- *Hawking* (1981): *"There ought to be something very special about the boundary conditions of the universe and what can be more special than the condition that there is no boundary"*

- Tunneling proposal (*Vilenkin*): creation of the universe seen as a regular tunneling event



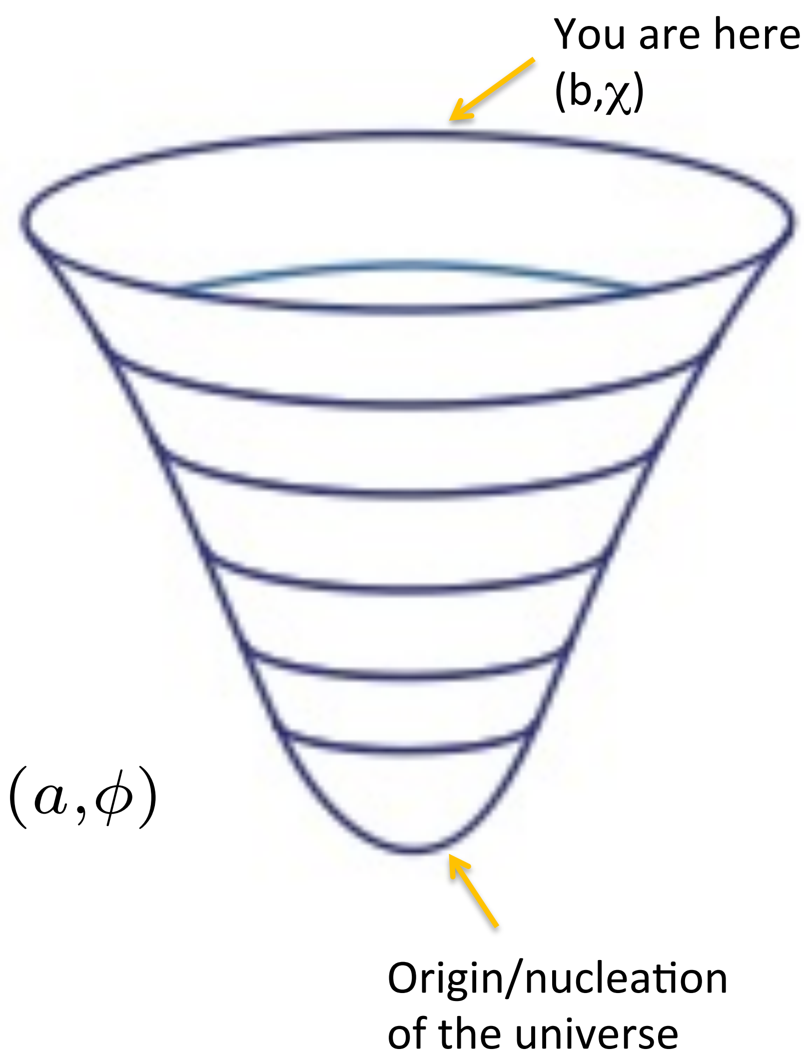
[Hartle & Hawking; Vilenkin]

No-boundary and tunneling proposals

- The big bang is then replaced by semi-classical closed (and regular?) geometries
- But how can we calculate this in practice?

$$\Psi(b,\chi) = \int_{\mathcal{C}} \mathcal{D}a\mathcal{D}\phi e^{-S_E(a,\phi)}$$

$$\approx e^{-S_{E,ext}(b,\chi)}$$



Two approaches

- Euclidean
  - In analogy with Wick rotation in QFT it was hoped that this would lead to better convergence
  - However conformal mode problem
- Lorentzian
  - No conformal mode problem
  - Causality can be built in
  - Not clear whether the path integral actually converges

[Hawking, Hartle, Gibbons, Perry,...]

[Vilenkin, Teitelboim,...]

$$S = \int \mathrm{d}t N \left(-3a\frac{\dot{a}^2}{N^2} + \frac{1}{2}a^3\frac{\dot{\phi}^2}{N^2} + \cdots \right)$$

Opposite signs

Gravity plus Cosmological Constant

- We will consider the simple system

$$\Psi = \int_{\mathcal{C}} \delta N \, \delta a \, e^{iS(N,a)/\hbar}$$

Can add ghosts and choose constant N gauge in a integral – see e.g. [Teitelboim]

with
$$S = \frac{1}{2} \int \mathrm{d}^4 x \sqrt{-g} \, (R - 2\Lambda)$$

- For a standard minisuperspace metric

$$\mathrm{d}s^2 = -N^2 \mathrm{d}t^2 + a^2 \mathrm{d}\Omega_3^2$$

this is hard to solve

$$S = 2\pi^2 \int \mathrm{d}t N \left(-3a\frac{\dot{a}^2}{N^2} + 3ka - a^3\Lambda \right)$$

An useful form of the metric

- Technically much simpler to consider

$$\mathrm{d}s^2 = -\frac{N^2}{q(t)}\mathrm{d}t^2 + q(t)\mathrm{d}\Omega_3^2$$

[Halliwell & Louko (1988)]

since then the action becomes quadratic

$$S = 2\pi^2 \int \mathrm{d}t \left(-\frac{3}{4N}\dot{q}^2 + 3kN - N\Lambda q \right)$$

- Then the integral over $q=a^2$ is simply a Gaussian, and can be done exactly

Path integral for the propagator

- We are left with an ordinary integral over the lapse function

$$G[q_1; q_0] = \frac{3\pi i}{2\hbar} \int_0^\infty \frac{\mathrm{d}N}{N^{1/2}} e^{2\pi^2 i S_0/\hbar}$$

Integrate only over N>0
-> causality and no
double counting
See [Teitelboim '80s]

$$S_0 = N^3 \frac{\Lambda^2}{36} + N\left(-\frac{\Lambda}{2}(q_0 + q_1) + 3\right) + \frac{1}{N}\left(-\frac{3}{4}(q_1 - q_0)^2\right)$$

$q_0=a_0^2$, initial value of scale factor $q_1=a_1^2$, final value of scale factor

- With canonical normalisation $M \equiv N^{1/2}$
- Asymptotic behaviour:

$$G \sim \int dM e^{i\left(M^6\Lambda^2 - \frac{q_1^2}{M^2}\right)/\hbar}$$

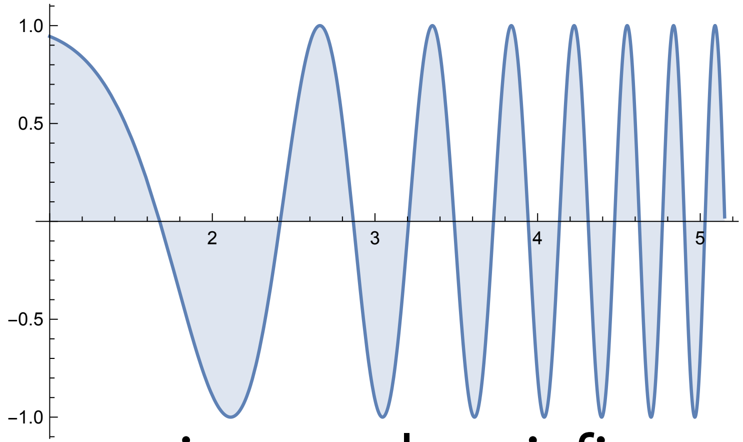
Lorentzian path integral converges!

- Leibniz convergence test for alternating sum:

$$a = \sum_{i=0}^{\infty} (-1)^i a_i \quad \text{converges if} \quad \begin{cases} a_{i+1} \le a_i \\ \lim_{i \to \infty} a_i = 0 \end{cases}$$

$$I = \int_0^\infty \mathrm{d}x \; e^{ix^n} \qquad n > 1 \text{ or } n < -1$$

$$\mathrm{Re}[I] = \int_0^\infty \mathrm{d}x \; \cos(x^n)$$



- Divide up as follows:
- We have n=6 and n=-2, hence our integral satisfies the test!
- *But what does it converge to?*

Picard-Lefschetz theory

- We are interested in oscillatory integrals, whose convergence properties are not clear, in particular the Feynman integral

$$\psi(x_f, x_i) = \int_{\mathcal{C}} \delta x \, e^{iS[x(t)]/\hbar}$$

- View the integrand $\mathcal{I} = iS/\hbar$ as a holomorphic function of $x \in \mathbb{C}$, then we might be able to find an appropriate convergent integration contour

Cf. Wick rotation where a coordinate is continued to the complex plane. But coordinates are not physical in GR, hence it seems preferable to continue the fields to the complex plane

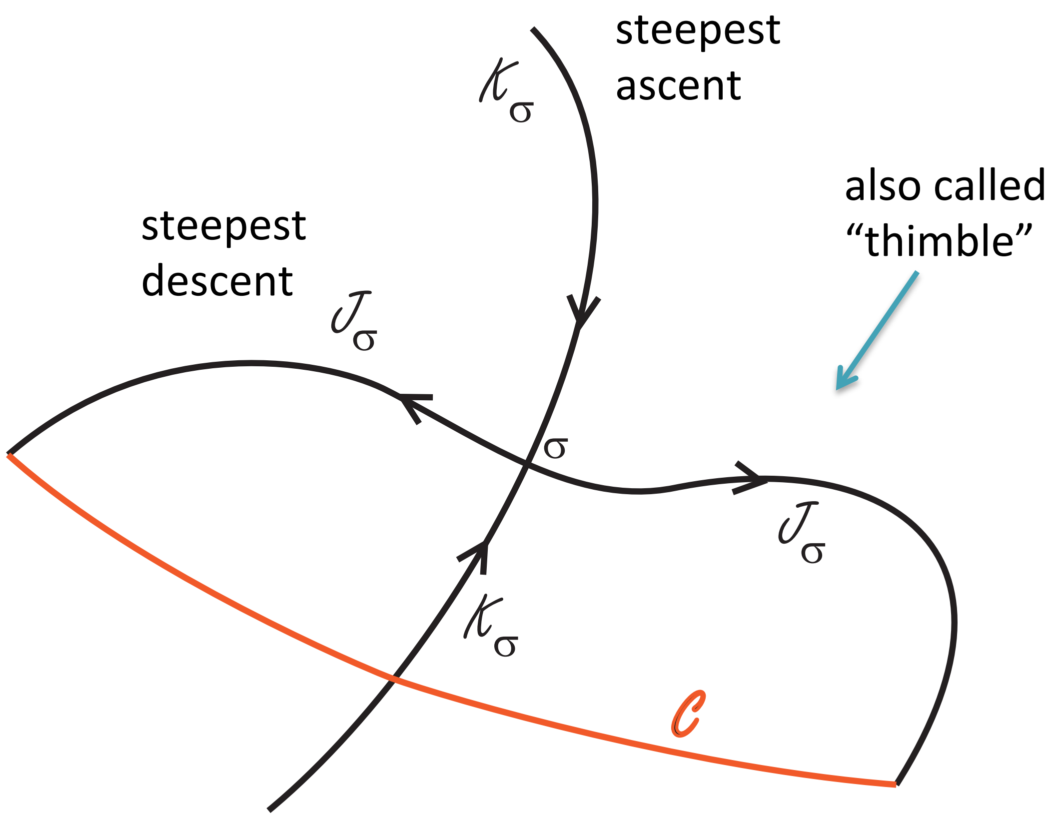
Picard-Lefschetz theory

- Cauchy's theorem tells us that a complex integration contour can be deformed
- Picard-Lefschetz theory tells us how it should be deformed

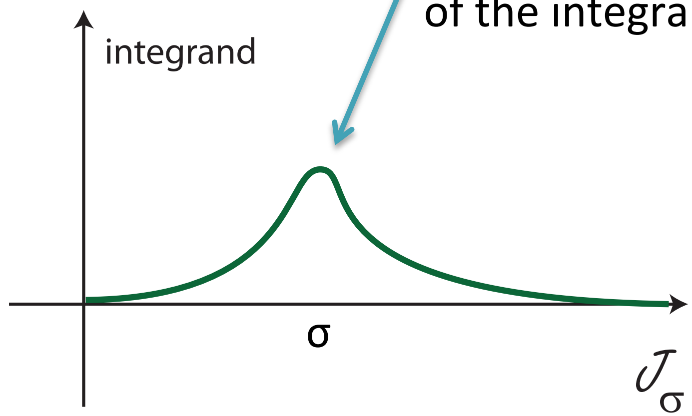
- A review is provided by E. Witten "Analytic continuation of Chern-Simons theory" (2010)

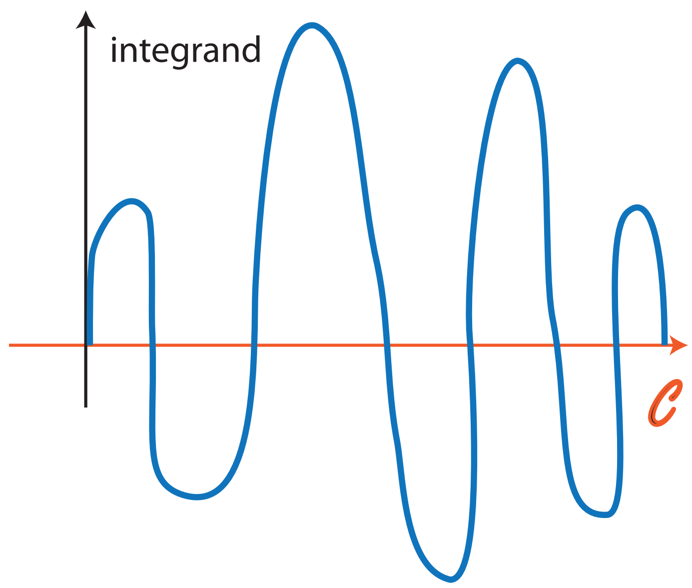
From conditionally to absolutely convergent

Look at the magnitude of the integrand:



Saddle point
of the integrand





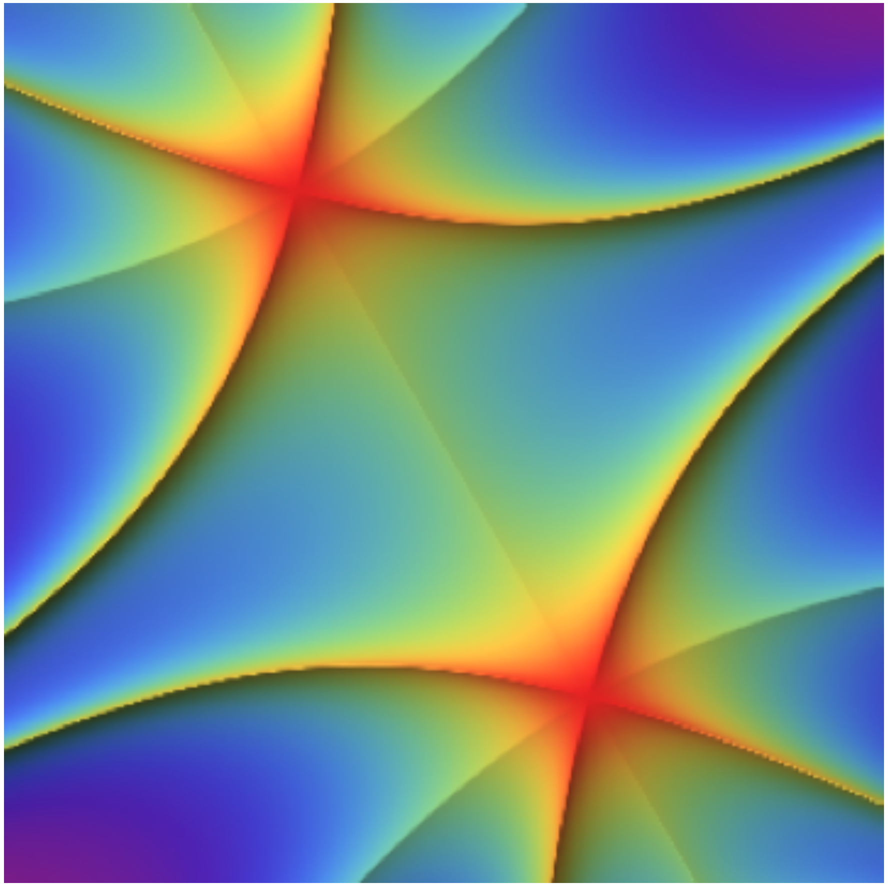
Picard-Lefschetz theory

- Which Lefschetz thimbles contribute? We would like to re-express the original integration contour as a sum over (relevant) thimbles: $$\mathcal{C} = \sum n_\sigma \mathcal{J}_\sigma$$
- Only those thimbles contribute which can be reached from the initial contour via downwards flow
- Final result:

$$\boxed{\int \mathrm{d}x \, e^{\mathcal{I}} = \sum_\sigma n_\sigma \, e^{i \, Im(\mathcal{I}_\sigma)} \int_{\mathcal{J}_\sigma} e^h}$$

Example

- Airy function $Ai(\phi) = \dfrac{1}{2\pi} \displaystyle\int dx e^{i\left(\frac{x^3}{3} + \phi x\right)}$
- Two saddle points at $x_s = \pm i\sqrt{\phi}$
- But do they both contribute? This depends on the argument:
- Example $Ai(e^{i\pi/3})$



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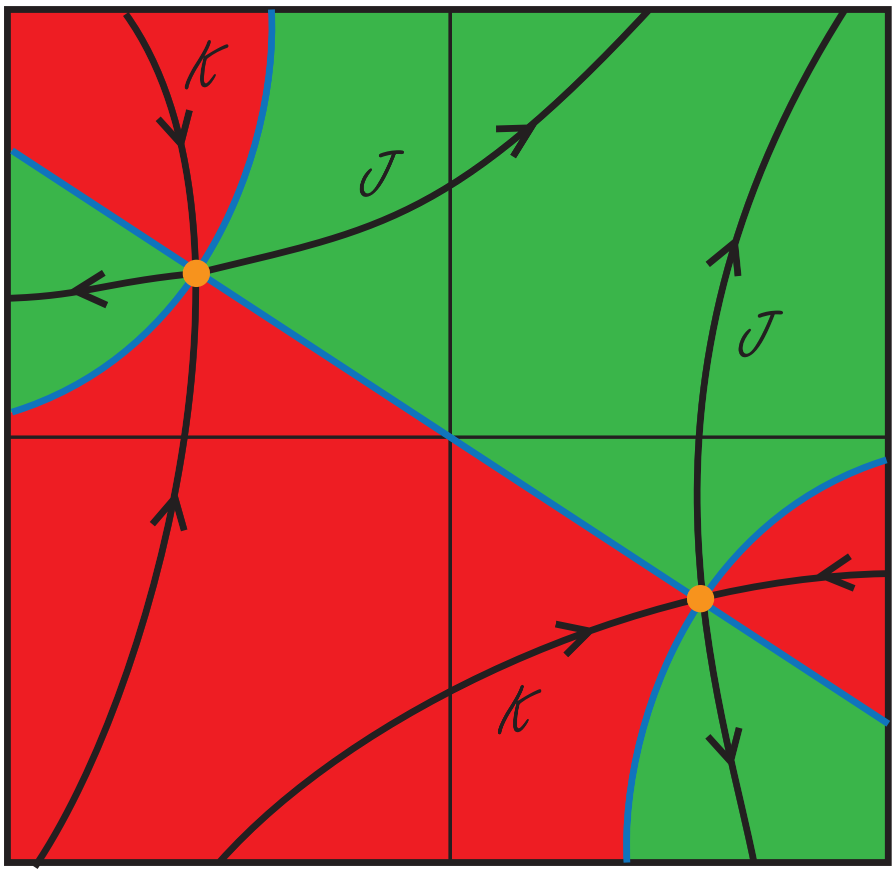

Green: integrand smaller than at saddle point
Red: integrand larger than at saddle point

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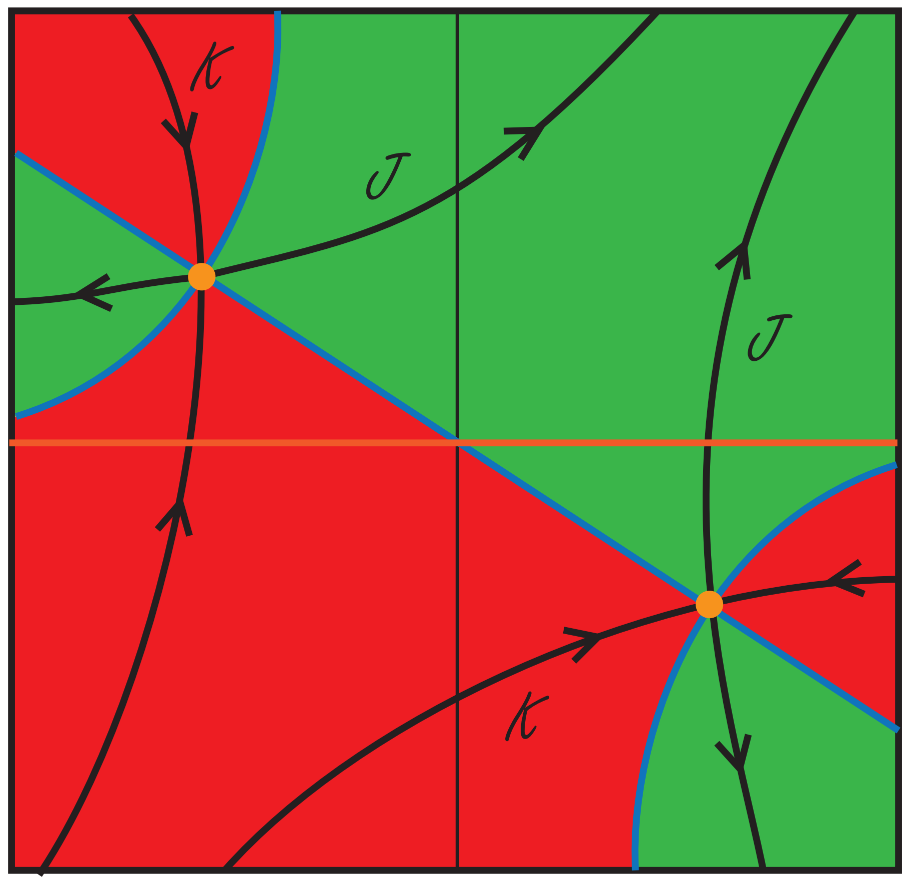

Assume defining contour is along real line, then flow downward

Example

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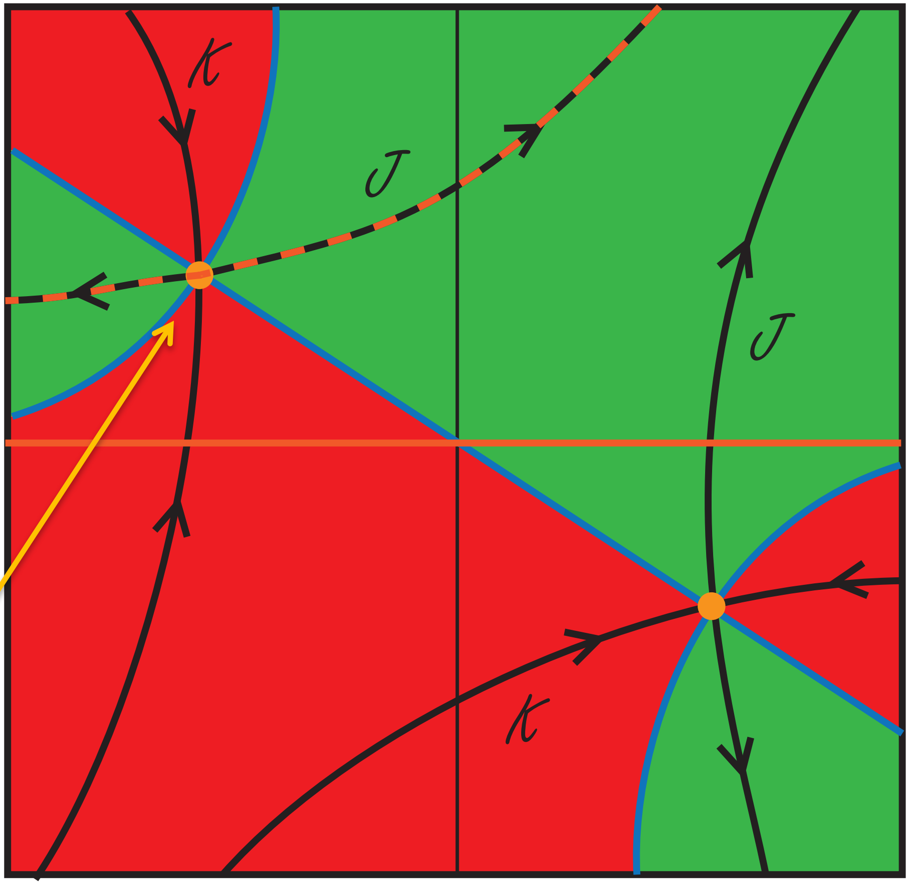


$$Ai(e^{i\pi/3}) \approx \frac{1}{2\pi} e^{i\left(\frac{x_s^3}{3} + e^{i\pi/3} x_s\right)}$$

with $x_s \to e^{7i\pi/6}$

Lapse integral

$$G[q_1;q_0]=\frac{3\pi i}{2\hbar}\int_0^\infty \frac{\mathrm{d}N}{N^{1/2}}e^{2\pi^2 iS_0/\hbar}$$

$$S_0=N^3\frac{\Lambda^2}{36}+N\left(-\frac{\Lambda}{2}(q_0+q_1)+3\right)+\frac{1}{N}\left(-\frac{3}{4}(q_1-q_0)^2\right)$$

- There are 4 saddle points:

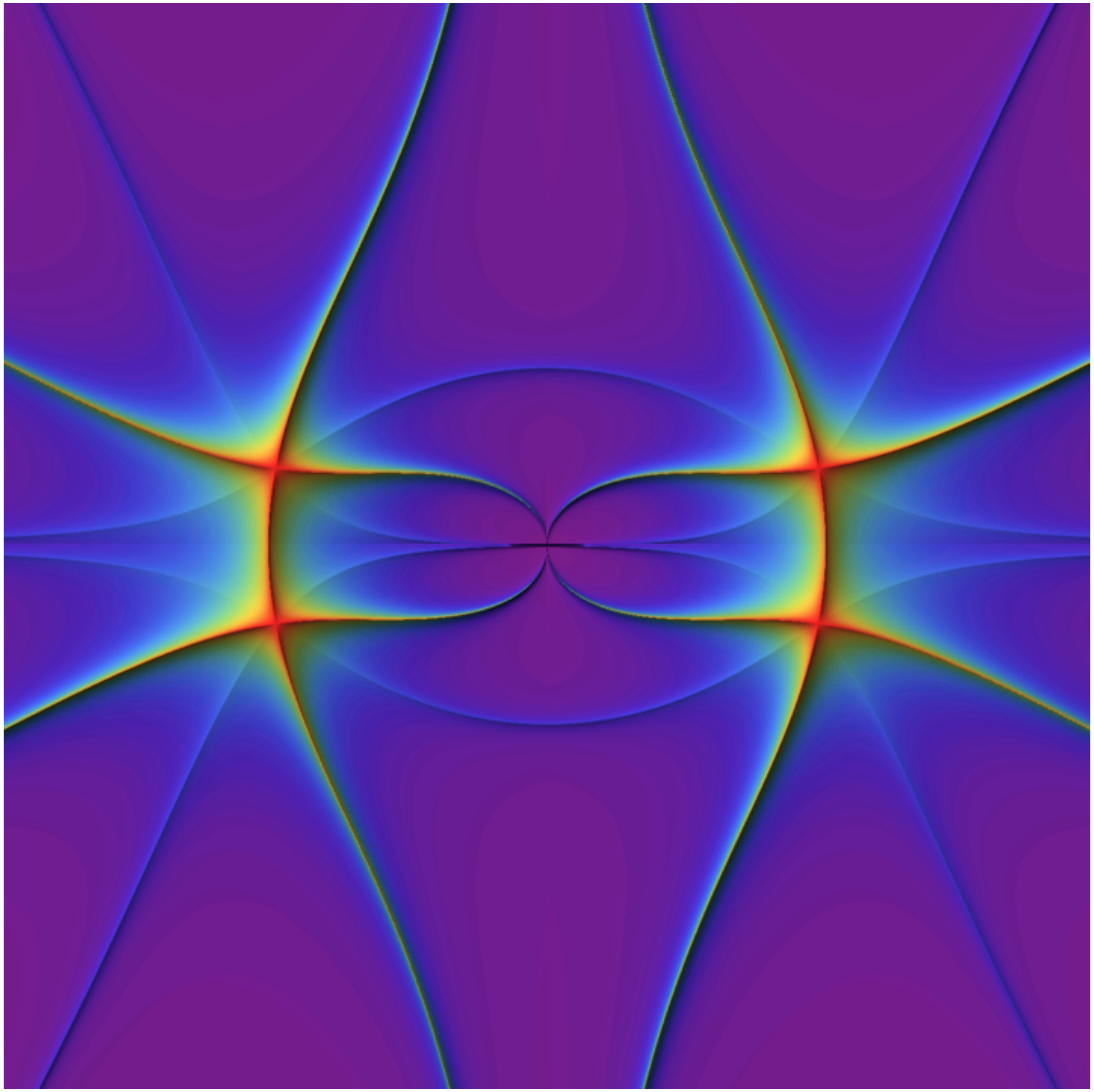
$$N_s=\pm\frac{3}{\Lambda}\left[(\frac{\Lambda}{3}q_0-1)^{1/2}\pm(\frac{\Lambda}{3}q_1-1)^{1/2}\right]$$

- The saddle points will be real/complex depending on the signs of $\Lambda q-3$
- Now we can apply Picard-Lefschetz theory

No-boundary condition q_0=0

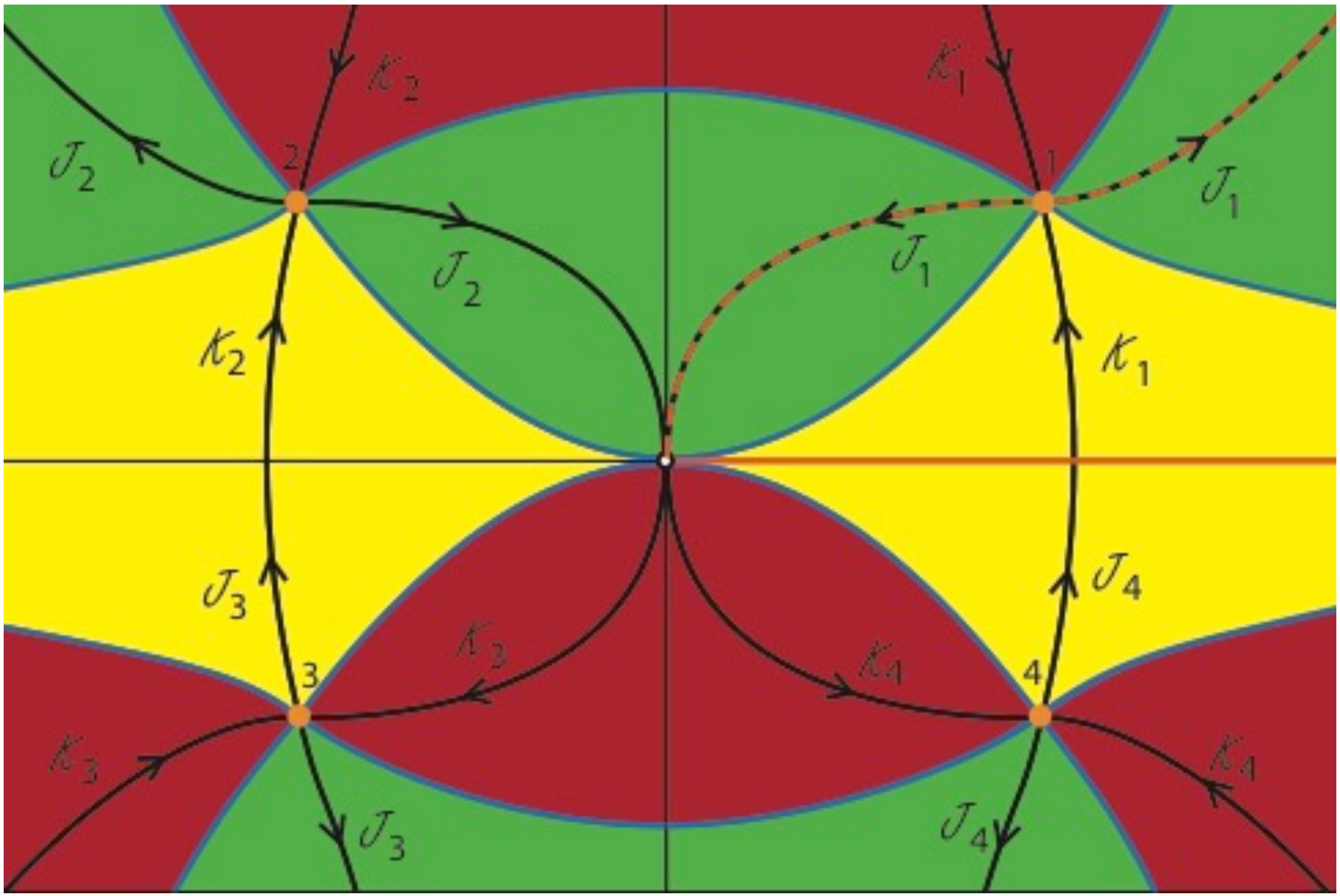
- Propagator from zero scale factor q_0=0 to a large final value q_1

- Saddle points are complex



No-boundary conditions

- Upward/downward flows:

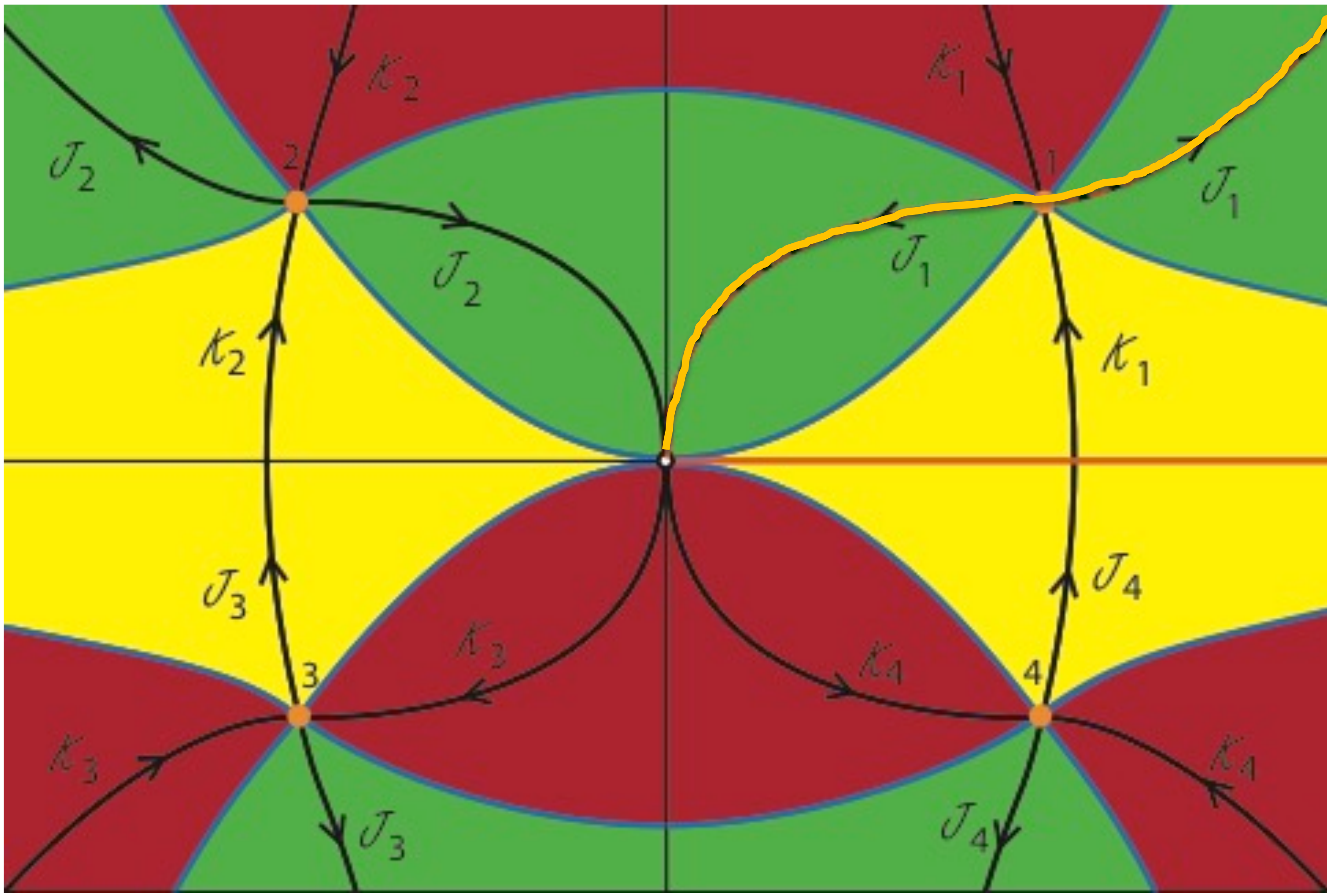


Real time contour

No-boundary conditions

- Upward/downward flows:

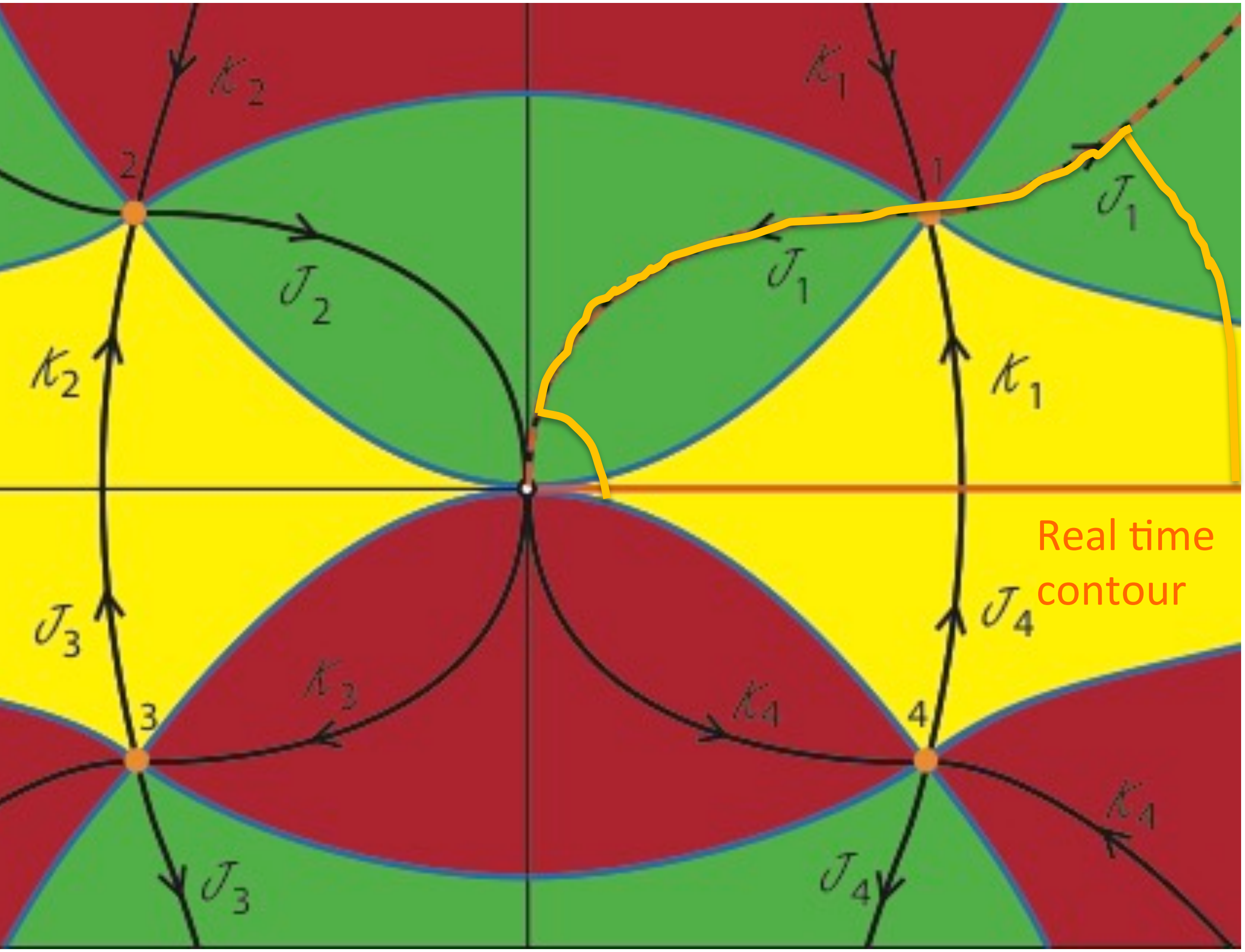
Only one Lefschetz thimble contributes



Real time contour

No-boundary conditions

- Convergence near zero/at infinity:



Near $N = 0$:

$$\int \mathrm{d}N e^{-\frac{i}{N}}$$

Near $N = \infty$:

$$\int \mathrm{d}N e^{iN^3}$$

Wavefunction for no-boundary conditions

- The wavefunction is dominated by a single saddle point, yielding

$$\boxed{\Psi_{nb}(q_1) \approx e^{i\frac{\pi}{4}} \frac{3^{1/4}}{2(\Lambda q_1 - 3)^{1/4}} e^{-12\pi^2/(\hbar\Lambda) - i4\pi^2\sqrt{\frac{\Lambda}{3}}(q_1 - \frac{3}{\Lambda})^{3/2}/\hbar}}$$

The weighting is inverse to that advocated by Hartle and Hawking, and is the same as for Vilenkin's tunneling wavefunction

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Picard-Lefschetz theory implies that relevant saddle points will always contribute with a suppressed amplitude – this makes sense physically as quantum processes are suppressed (and not enhanced) compared to classical evolution

Include Tensor Perturbations

If we add perturbations, the propagator is given by

$$\Psi[q_1, \phi_1; q_0, \phi_0] = \int_{0^+}^{\infty} \mathrm{d}N \int \mathcal{D}q \int \mathcal{D}\phi \, e^{iS[q,\phi,N]/\hbar}$$

with $S = S^{(0)} + S^{(2)}$

where the perturbation action is (e.g. for a gravity wave mode with wavenumber l)

$$S^{(2)} = \frac{1}{2} \int N_s \mathrm{d}t \, \mathrm{d}^3 x \left[q^2 \left(\frac{\dot{\phi}}{N_s} \right)^2 - l(l+2)\phi^2 \right]$$

$$= \frac{1}{2} \left[\frac{q^2}{N_s} \phi \dot{\phi} \right]_0^1 \quad \text{(on-shell)}$$

Include Tensor Perturbations

- In physical time,

$$S^{(2)} = \frac{1}{2}\int N \mathrm{d}t_p\, \mathrm{d}^3x \left[a^3 \left(\frac{\phi_{,t_p}}{N}\right)^2 - a\, l(l+2)\phi^2\right]$$

- Solution to the equation of motion (at background saddle point), with $H = \sqrt{\frac{3}{\Lambda}}$

$$\phi = c_1 \left(1 + \frac{i}{\sinh(Ht_p)}\right)^{\frac{l}{2}} \left(1 - \frac{i}{\sinh(Ht_p)}\right)^{-\frac{l+2}{2}} \left(1 - \frac{i(l+1)}{\sinh(Ht_p)}\right)$$
$$+ c_2 \left(1 - \frac{i}{\sinh(Ht_p)}\right)^{\frac{l}{2}} \left(1 + \frac{i}{\sinh(Ht_p)}\right)^{-\frac{l+2}{2}} \left(1 + \frac{i(l+1)}{\sinh(Ht_p)}\right)$$

- For P-L instanton, at South Pole $\sinh(Ht) = -i$
- Then regularity implies $c_2=0$ (now call $c_1 = \phi_1$)

- The action then becomes

$$\Psi \propto e^{\phi_1^2 \frac{l(l+2)}{2\hbar H^2}(-i\sinh(Ht_p)+l+1)}$$

- so that the weighting is given by

$$\boxed{|\Psi_\phi| \approx e^{+\phi_1^2 \frac{l(l+1)(l+2)}{2\hbar H^2}}}$$

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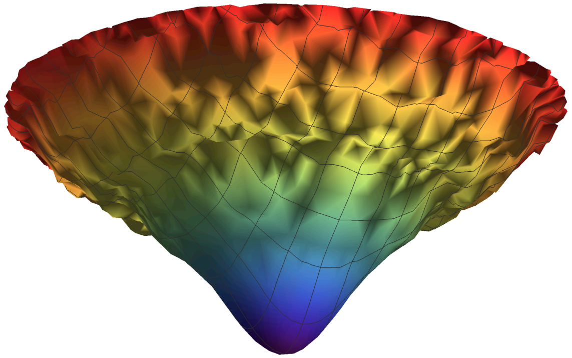
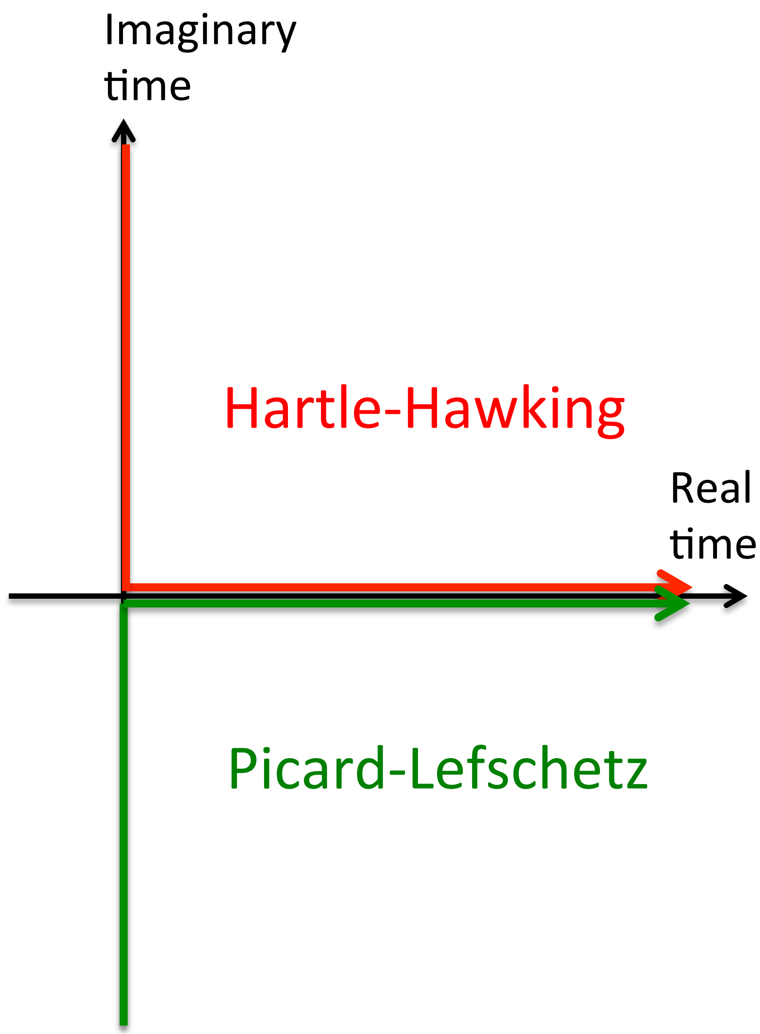


- Thus the perturbations obey an *inverse Gaussian* distribution – the distribution prefers large fluctuations and the model breaks down!

Analogy in terms of Wick rotation

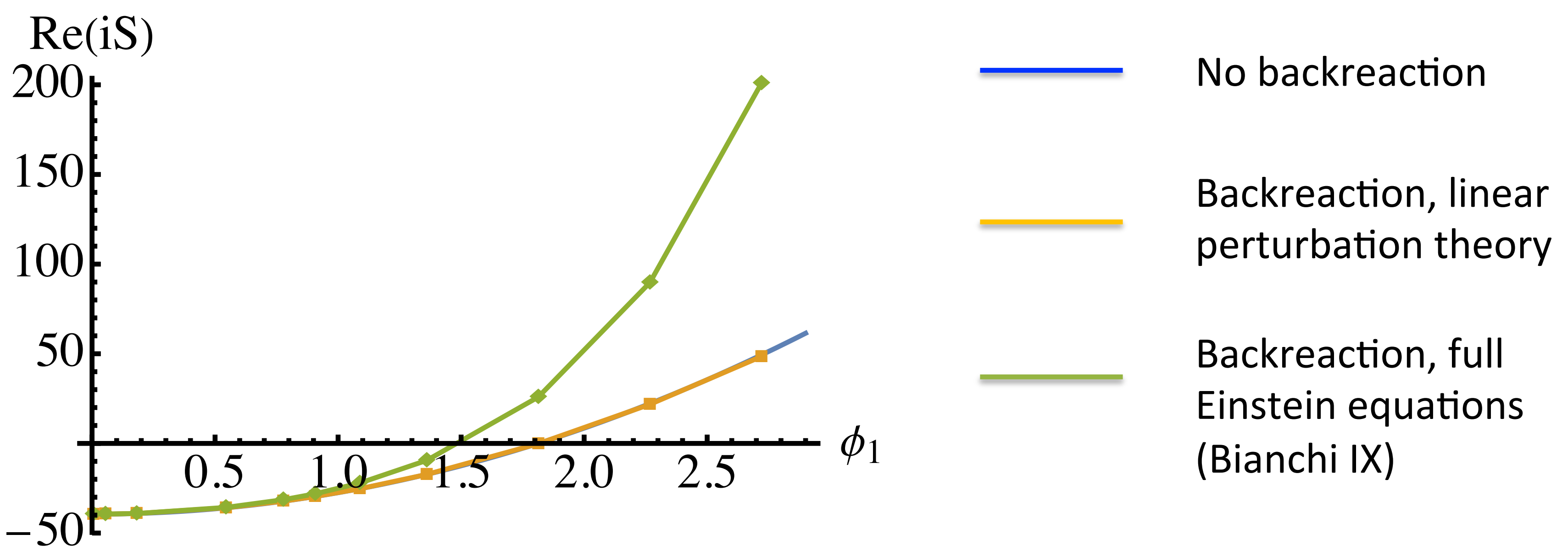
One way to understand this result is to realize that Picard-Lefschetz theory forces one to choose the "wrong" Wick rotation

But note: here the Wick rotation arose from analytic continuation of the *fields*, not the time coordinate

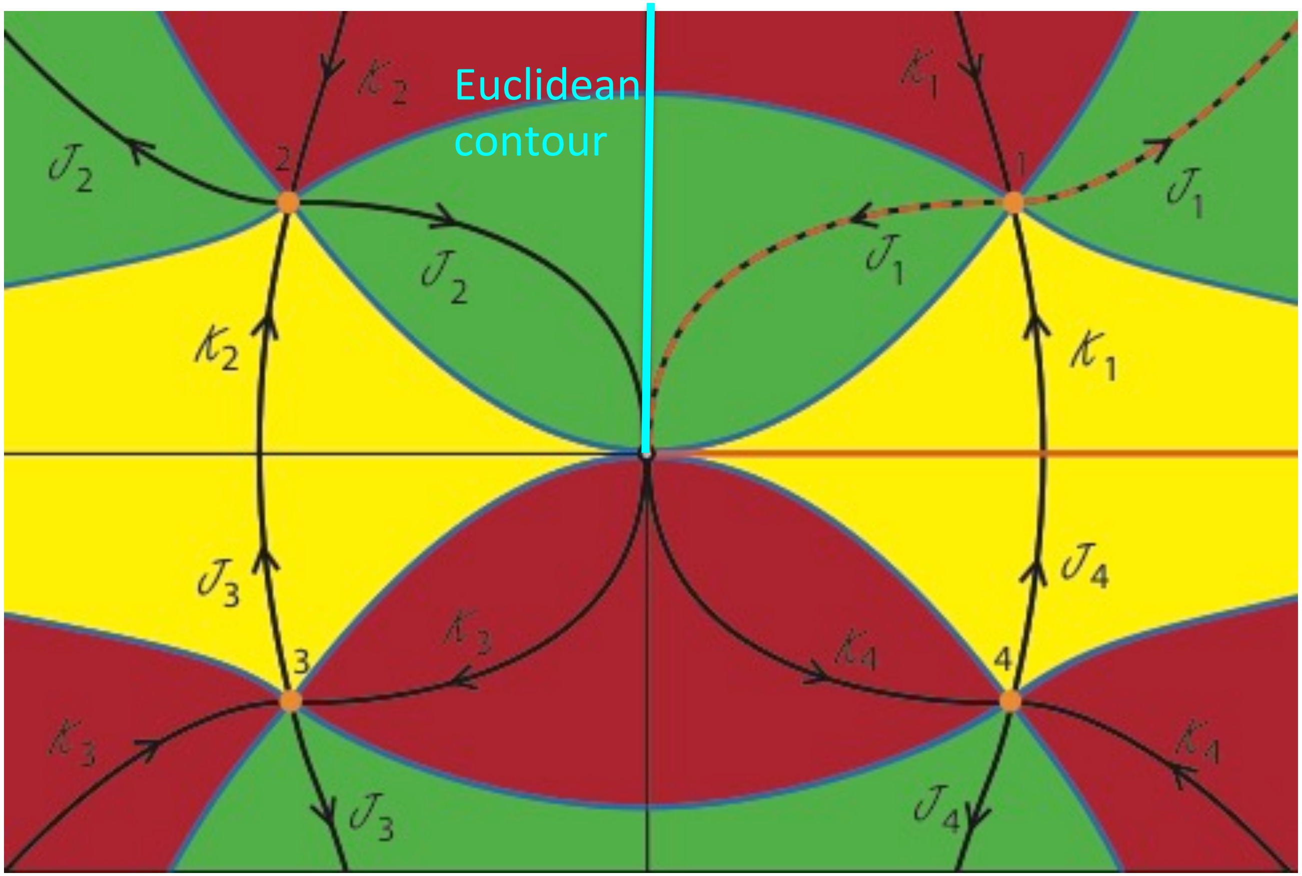


Do our approximations break down?

- Backreaction (i.e. corrections to the scale factor due to the linear perturbations) change the results very little
- Have also checked that the full non-linear l=2 modes show the same qualitative behavior – the instability in fact becomes even stronger at the saddle points



No Euclidean path integral!



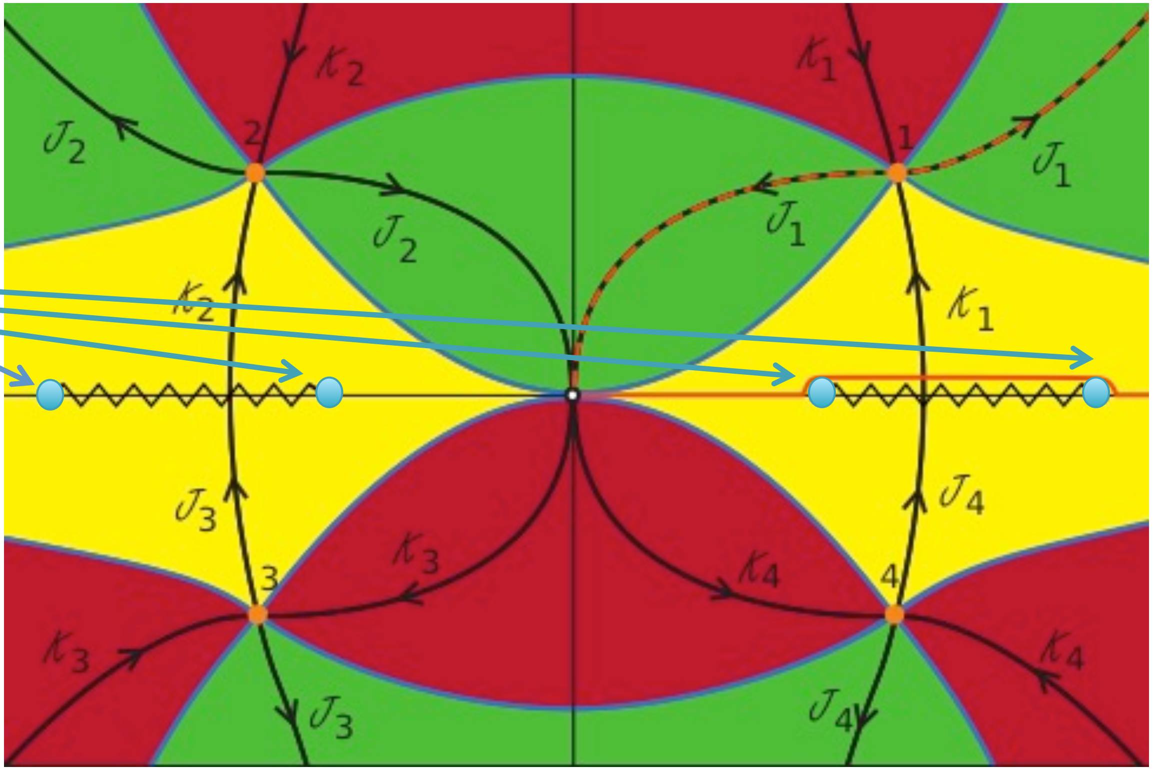
- The Euclidean path integral cannot be approximated by the saddle point method, and is simply not well-defined

- Other, inherently complex contours have been proposed by Diaz Dorronsoro et al., but (in our view) they lead to *inconsistencies*

[Diaz Dorronsoro, Halliwell, Hartle, Hertog, Janssen, Vreys: 1705.05340 & 1804.01102]

Properties of the perturbed action

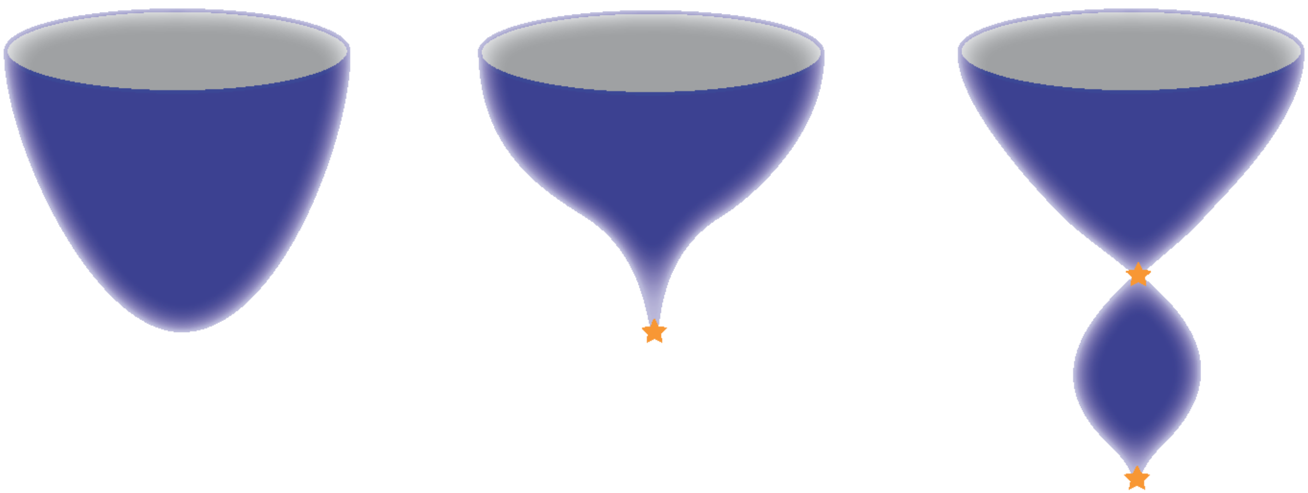
- The off-shell action has branch points, i.e. it is not analytic



- This comes about because the integral over perturbations is an infinite dimensional integral

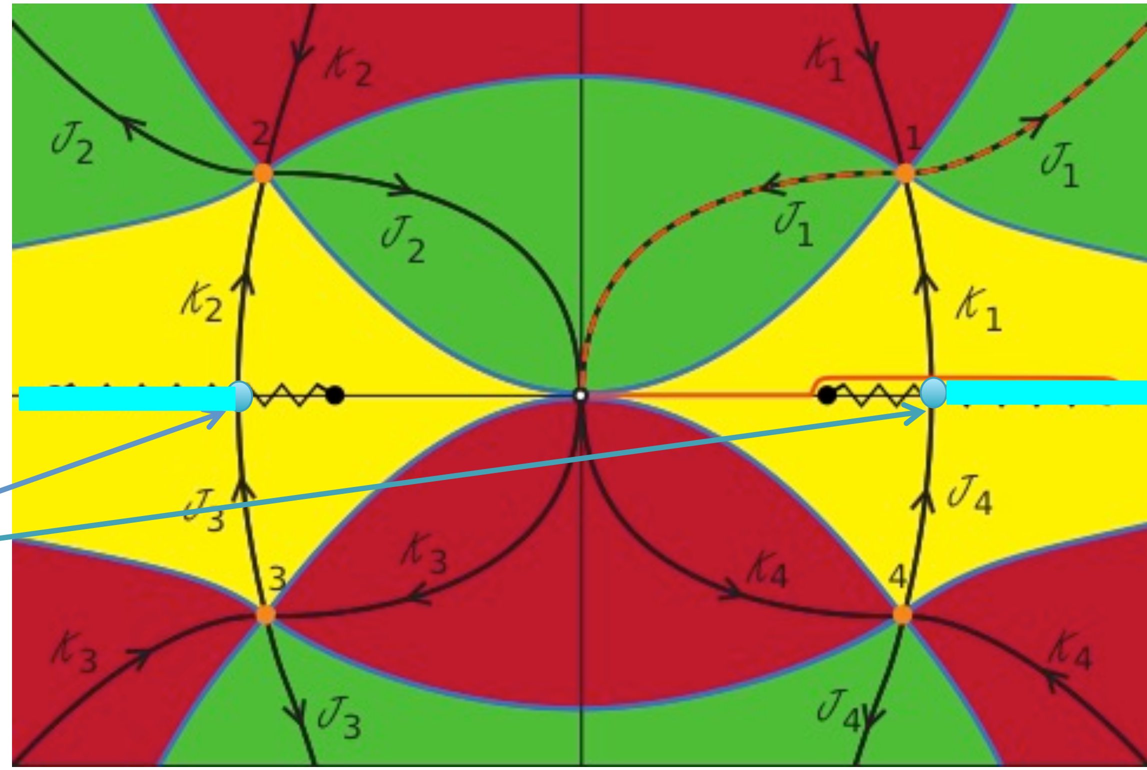
Strong singularities on real N line

- In fact, the behavior on the real N line is worse at large N:



- The off-shell geometries develop first one, then two singularities at which the perturbative action blows up

Properties of the perturbed action



$$N_\star = \pm\sqrt{\frac{3q_1}{\Lambda}}$$

- Must exclude real N line for $|N|>N_*$, since the perturbative action is not defined on those half-lines
- Thimbles are not affected

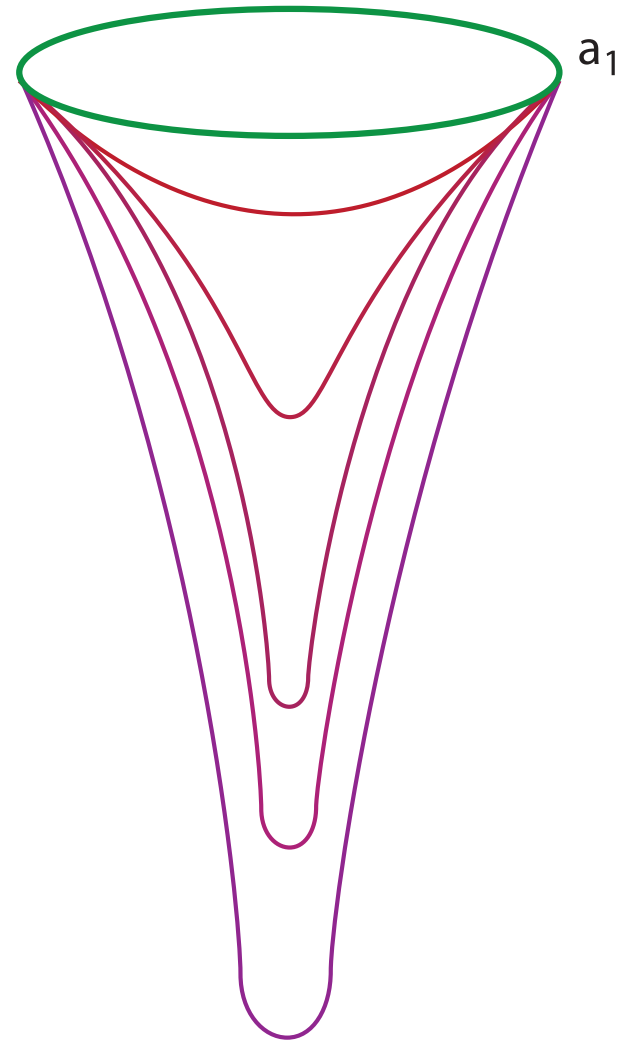
Regular Geometries

- One could sum over only manifestly regular geometries, i.e. where there are no off-shell singularities at all!
- Simplest model: sum over (complexified) spheres,

$$a(t) = \pm r \sin\left(\frac{N_E\, t}{r}\right)$$

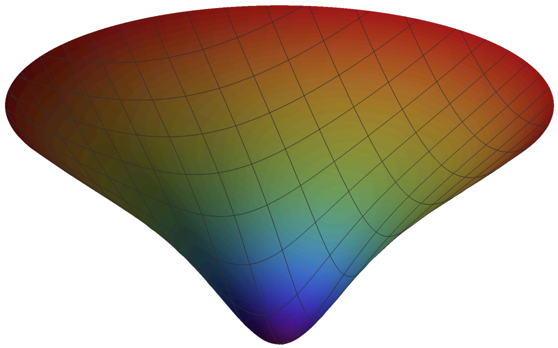
plus regular perturbations on the spheres

- This leads to the same instability
- Backreaction can be checked to be very small



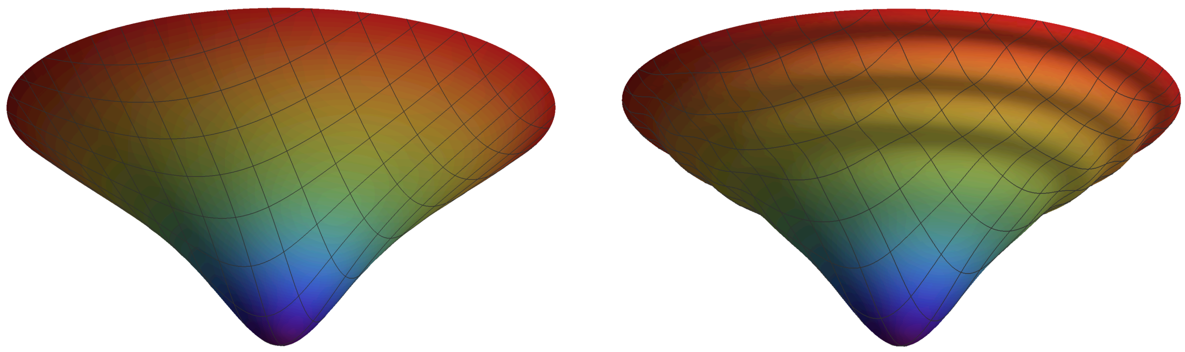
[Halliwell & Louko (1989); Di Tucci & JLL (2018)]

Main lessons



- With $\Lambda > 0$ the Lorentzian path integral for gravity exists! (at least in minisuperspace)

Main lessons



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- With Λ>0 the Euclidean path integral for gravity does *not* exist

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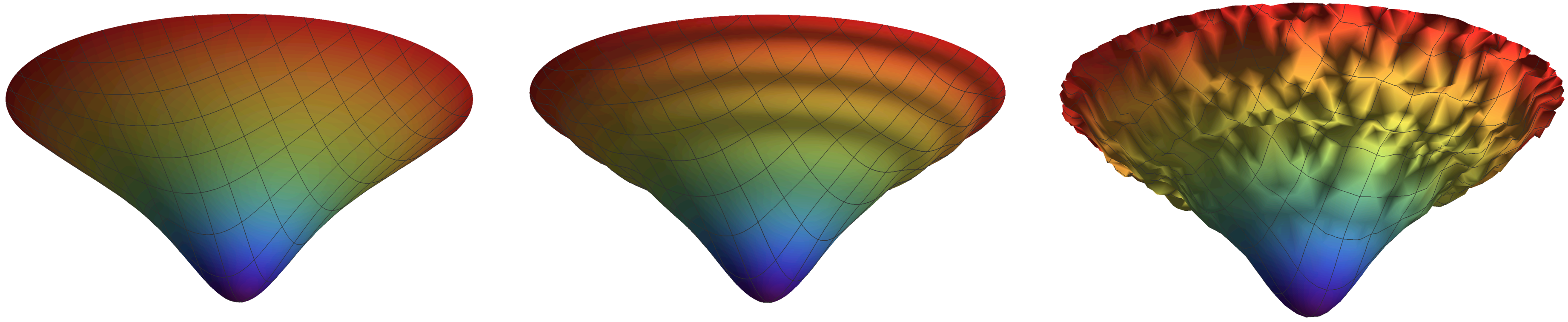

- With $\Lambda > 0$ the Lorentzian path integral for gravity exists! (at least in minisuperspace)
- With $\Lambda > 0$ the Euclidean path integral for gravity does *not* exist
- The question of initial conditions remains wide open!

Outlook & applications

- Are there models (with a different matter content & different boundary conditions) where there is no background/perturbations mismatch?

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- Are there models (with a different matter content & different boundary conditions) where there is no background/perturbations mismatch?

- Applications of these methods:
  - When is QFT in curved spacetime a good approximation? What about the beginning of an inflationary phase?
  - Can we describe quantum transitions through the big bang with these methods?
  - Infinite dimensional Picard-Lefschetz might allow for a better mathematical definition of Lorentzian path integrals

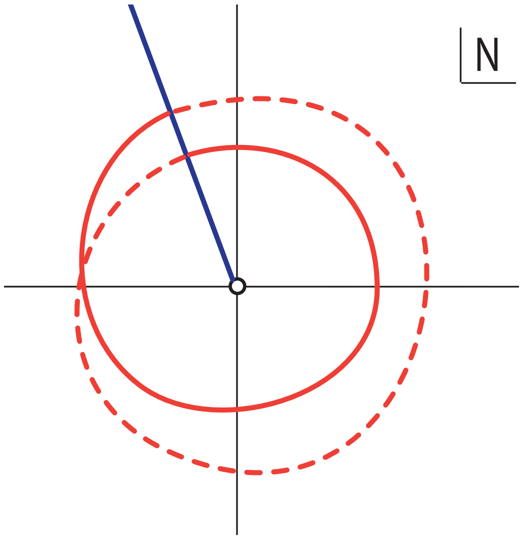
Inconsistency of the new "circular" no-boundary proposal – look at isotropic boundary conditions

Bianchi IX metric:

$$-\frac{N^2}{q}dt^2 + \frac{p}{4}(\sigma_1^2 + \sigma_2^2) + \frac{q}{4}\sigma_3^2$$

Leads to path integral:

$$\int \frac{\mathrm{d}N}{N} e^{iS^{BianchiIX}/\hbar}$$



Isotropic metric:

$$-\frac{N^2}{q}dt^2 + \frac{q}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

Leads to path integral:

$$\int \frac{\mathrm{d}N}{\sqrt{N}} e^{iS^{iso}/\hbar}$$

[Diaz Dorronsoro, Halliwell, Hartle, Hertog, Janssen & Vreys, 1804.01102]

[Feldbrugge, JLL & Turok, 1805.01609]

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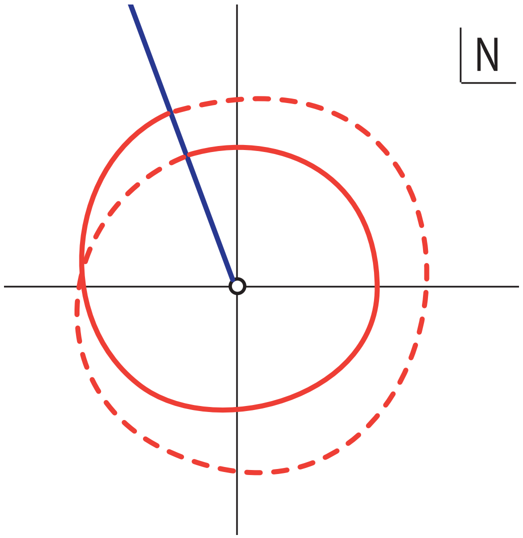

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Leads to path integral:

$$\int \frac{\mathrm{d}N}{\sqrt{N}} e^{iS^{iso}/\hbar}$$

- Because of the $N^{-1/2}$ prefactor, one must choose a branch cut and the integral has to wind around the origin twice to obtain a closed contour – this integral is zero!
- But we were describing the same physical situation, up to an extra integral over deformations of the sphere. To leading semiclassical order the results ought to agree, but they don't – hence the circular contour is inconsistent!
- Similar inconsistency for any change of dimension by one