INSTITUT D'ASTROPHYSIQUE DE PARIS
Unité mixte de recherche 7095 CNRS - Université Pierre et Marie Curie

Fifteenth Marcel Grossmann Meeting
Rome

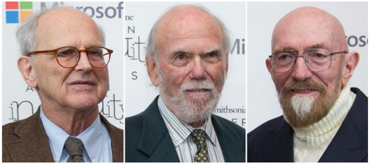
POST-NEWTONIAN THEORY
&
GRAVITATIONAL WAVES

Luc Blanchet

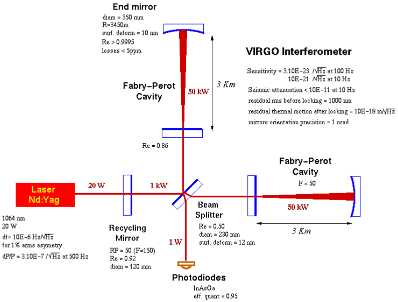
Gravitation et Cosmologie ($\mathcal{G}\mathbb{R}\varepsilon\mathbb{C}\mathcal{O}$)
Institut d'Astrophysique de Paris

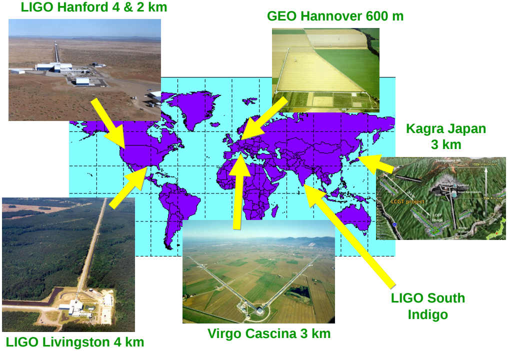
5 juillet 2018

World-wide network of gravitational wave detectors

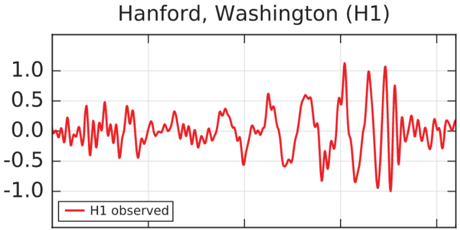


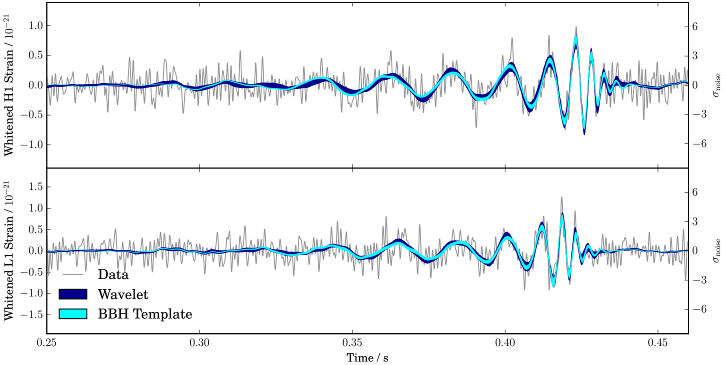
[Rainer Weiss, Barry Barish & Kip Thorne, Nobel prize 2017]



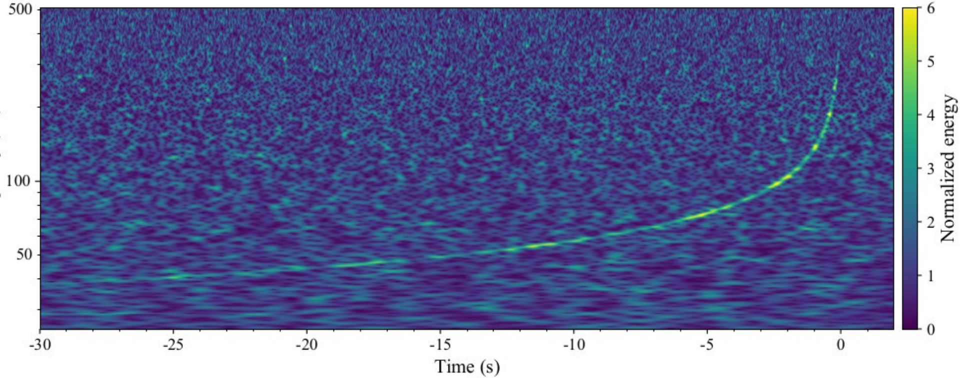


Binary black-hole event GW150914 [LIGO/VIRGO collaboration 2016]



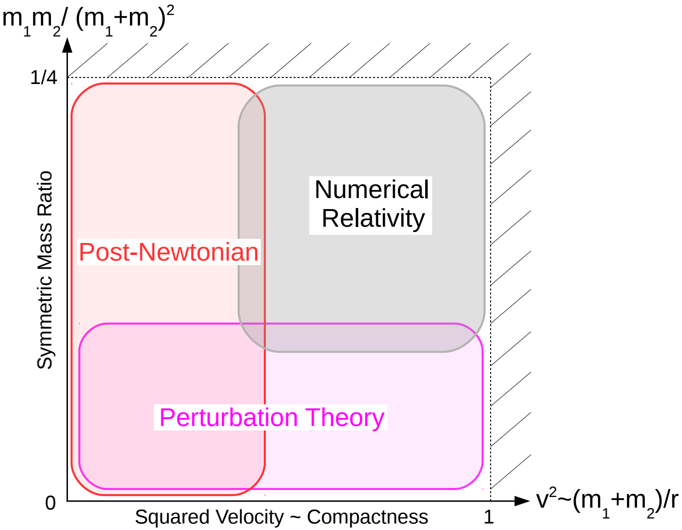


Binary neutron star event GW170817 [LIGO/Virgo 2017]

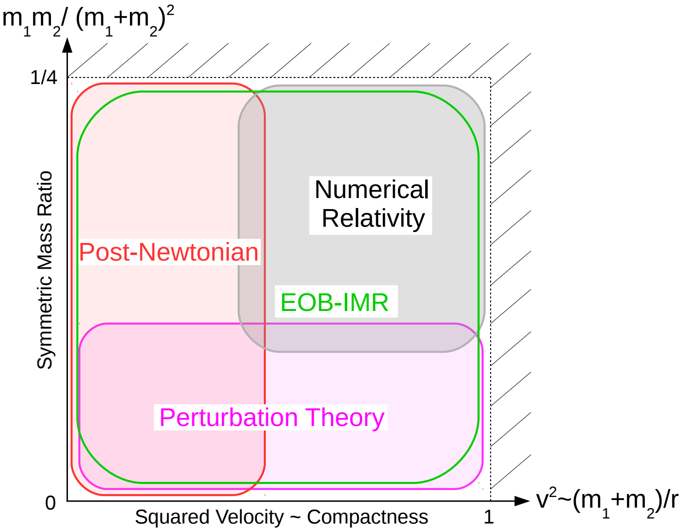


- The signal is observed during $\sim 100\,\mathrm{s}$ and ~ 3000 cycles and is the loudest gravitational-wave signal yet observed with a combined SNR of 32.4
- The chirp mass is accurately measured to $\mathcal{M} = \mu^{3/5} M^{2/5} = 1.98\, M_\odot$
- The distance is measured from the gravitational signal as $D = 40$ Mpc

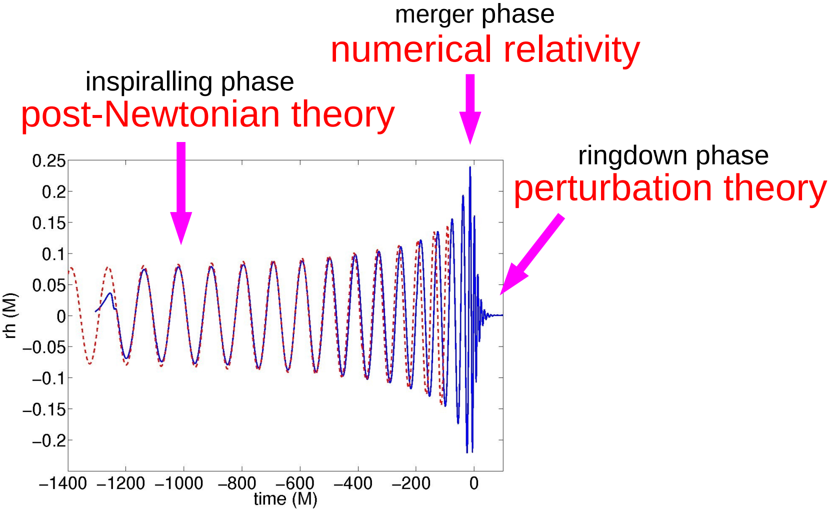
Methods to compute GW templates



Methods to compute GW templates



The gravitational chirp of compact binaries

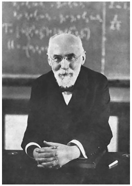


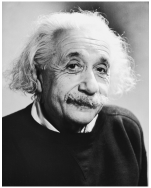
The GW templates of compact binaries

1. In principle, the templates are obtained by matching together:
   - A high-order 3.5PN waveform for the inspiral [Blanchet *et al.* 1998, 2002, 2004]
   - A highly accurate numerical waveform for the merger and ringdown [Pretorius 2005; Baker *et al.* 2006; Campanelli *et al.* 2006; Hannam, Husa, Sperhake *et al.* 2008]
2. In practice, for black hole binaries (such as GW150914), effective methods that interpolate between the PN and NR play a key role in the data analysis
   - Hybrid inspiral-merger-ringdown (IMR) waveforms are constructed by matching the PN and NR waveforms in a time interval through an intermediate phenomenological phase [Ajith, Hannam, Husa *et al.* 2011]
   - Effective-one-body (EOB) waveforms are based on resummation techniques extending the domain of validity of the PN approximation beyond the inspiral phase [see the review talk by Alessandro Nagar in the parallel session BN6]
3. In the case of neutron star binaries (such as GW170817), the templates are entirely based on the 3.5PN waveform

The 1PN equations of motion

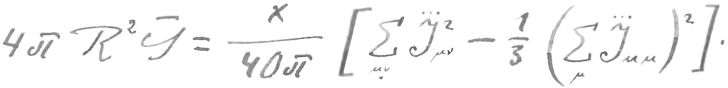
[Lorentz & Droste 1917; Einstein, Infeld & Hoffmann 1938]





$$
\boxed{
\begin{aligned}
\frac{\mathrm{d}^2\boldsymbol{r}_A}{\mathrm{d}t^2} = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}\boldsymbol{n}_{AB}\Bigg[1 - 4\sum_{C\neq A}\frac{Gm_C}{c^2 r_{AC}} - \sum_{D\neq B}\frac{Gm_D}{c^2 r_{BD}}\left(1 - \frac{\boldsymbol{r}_{AB}\cdot\boldsymbol{r}_{BD}}{r_{BD}^2}\right) \\
+ \frac{1}{c^2}\left(\boldsymbol{v}_A^2 + 2\boldsymbol{v}_B^2 - 4\boldsymbol{v}_A\cdot\boldsymbol{v}_B - \frac{3}{2}(\boldsymbol{v}_B\cdot\boldsymbol{n}_{AB})^2\right)\Bigg] \\
+ \sum_{B\neq A}\frac{Gm_B}{c^2 r_{AB}^2}\boldsymbol{v}_{AB}[\boldsymbol{n}_{AB}\cdot(3\boldsymbol{v}_B - 4\boldsymbol{v}_A)] - \frac{7}{2}\sum_{B\neq A}\sum_{D\neq B}\frac{G^2 m_B m_D}{c^2 r_{AB} r_{BD}^3}\boldsymbol{n}_{BD}
\end{aligned}
}
$$

Quadrupole moment formalism [Einstein 1918; Landau & Lifchitz 1947]



1. Einstein quadrupole formula

$$\left(\frac{\mathrm{d}E}{\mathrm{d}t}\right)^{\text{GW}} = \frac{G}{5c^5}\left\{\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} + \mathcal{O}\left(\frac{v}{c}\right)^2\right\}$$

2. Amplitude quadrupole formula

$$h_{ij}^{\text{TT}} = \frac{2G}{c^4 R}\left\{\frac{\mathrm{d}^2 Q_{ij}}{\mathrm{d}t^2}\left(t-\frac{R}{c}\right) + \mathcal{O}\left(\frac{v}{c}\right)\right\}^{\text{TT}} + \mathcal{O}\left(\frac{1}{R^2}\right)$$

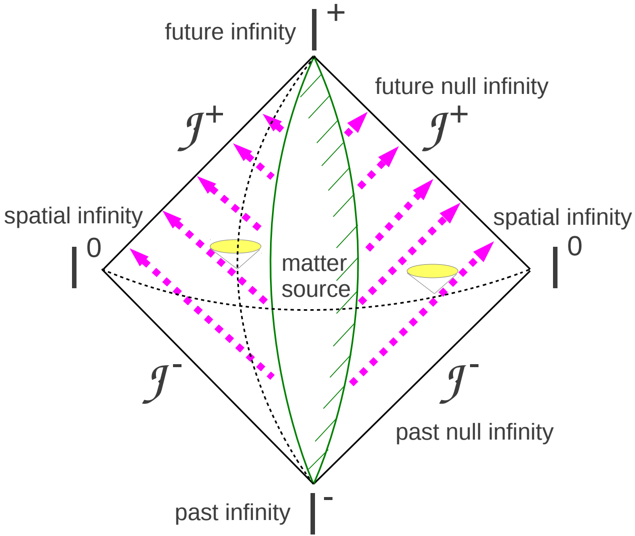
3. Radiation reaction formula [Chandrasekhar & Esposito 1970; Burke & Thorne 1970]

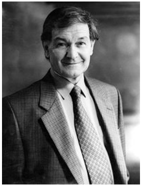
$$F_i^{\text{reac}} = -\frac{2G}{5c^5}\rho\, x^j \frac{\mathrm{d}^5 Q_{ij}}{\mathrm{d}t^5} + \mathcal{O}\left(\frac{v}{c}\right)^7$$

which is a 2.5PN $\sim (v/c)^5$ effect in the source's equations of motion

Asymptotic structure of radiating space-time

[Bondi-Sachs-Penrose formalism 1960s]





Multipolar-post-Minkowskian expansion

[Blanchet-Damour-Iyer formalism 1980-1990s]

1. Starts with the most general solution of the linearized equations outside an isolated source in the form of multipole expansions [Thorne 1980]
2. An explicit MPM algorithm is constructed out of it by induction at any order n in the post-Minkowskian expansion

$$h^{\mu\nu}_{\text{MPM}} = \sum_{n=1}^{+\infty} G^n \underbrace{h^{\mu\nu}_{(n)}[M_L, S_L]}_{\text{explicit functional of multipole moments}}$$

3. A finite-part (FP) regularization based on analytic continuation is required in order to cope with the divergency of the multipolar expansion when $r \to 0$

Multipolar-post-Minkowskian expansion

[Blanchet-Damour-Iyer formalism 1980-1990s]

Theorem 1:
The MPM solution is the most general solution of Einstein's vacuum equations outside an isolated matter system

Theorem 2:
The general structure of the PN expansion is

$$h^{\alpha\beta}_{\mathsf{PN}}(\mathbf{x},t,c) = \sum_{\substack{p\geqslant 2\\ q\geqslant 0}} \frac{(\ln c)^q}{c^p} h^{\alpha\beta}_{p,q}(\mathbf{x},t)$$

Theorem 3:
The MPM solution is asymptotically simple at future null infinity in the sense of Penrose [1963, 1965] and agrees with the Bondi-Sachs [1962] formalism

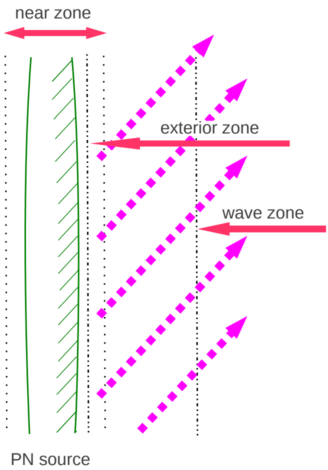
$$\underbrace{M_{\mathsf{B}}(u)}_{\text{Bondi mass}} = \underbrace{M}_{\text{ADM mass}} - \frac{G}{5c^5}\int_{-\infty}^{u} \mathrm{d}\tau\, M^{(3)}_{ij}(\tau) M^{(3)}_{ij}(\tau)$$

$$+ \quad \text{higher multipoles and higher PM computable to any order}$$

The MPM-PN formalism

[Blanchet 1995, 1998; Poujade & Blanchet 2002; Blanchet, Faye & Nissanke 2005]

A multipolar post-Minkowskian (MPM) expansion in the exterior zone is matched to a general post-Newtonian (PN) expansion in the near zone



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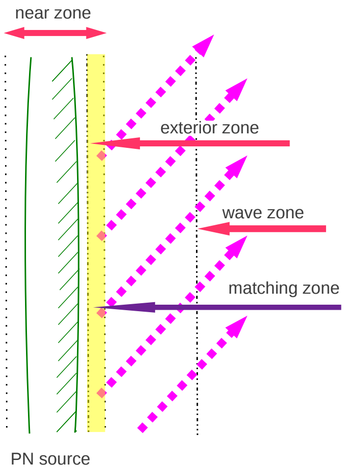


$$\underbrace{\overline{\mathcal{M}(h^{\mu\nu})} = \mathcal{M}(\bar{h}^{\mu\nu})}_{\text{matching equation}}$$

3.5PN energy flux of compact binaries

$$\mathcal{F}^{\mathrm{GW}} = \frac{32c^5}{5G}\nu^2 x^5 \Bigg\{ 1 + \overbrace{\left(-\frac{1247}{336} - \frac{35}{12}\nu\right) x}^{\text{1PN}} + \overbrace{4\pi x^{3/2}}^{\text{1.5PN tail}}$$

$$+ \left(-\frac{44711}{9072} + \frac{9271}{504}\nu + \frac{65}{18}\nu^2\right) x^2 + \overbrace{\left(-\frac{8191}{672} - \frac{583}{24}\nu\right)\pi x^{5/2}}^{\text{2.5PN tail}}$$

$$+ \Bigg[\frac{6643739519}{69854400} + \overbrace{\frac{16}{3}\pi^2 - \frac{1712}{105}\gamma_{\mathrm{E}} - \frac{856}{105}\ln(16\,x)}^{\text{3PN tail-of-tail}}$$

$$+ \left(-\frac{134543}{7776} + \frac{41}{48}\pi^2\right)\nu - \frac{94403}{3024}\nu^2 - \frac{775}{324}\nu^3 \Bigg] x^3$$

$$+ \underbrace{\left(-\frac{16285}{504} + \frac{214745}{1728}\nu + \frac{193385}{3024}\nu^2\right)\pi x^{7/2}}_{\text{3.5PN tail}} + \mathcal{O}\left(\frac{1}{c^8}\right) \Bigg\}$$

[see the talk by Tanguy Marchand in the parallel session BN6 for the term at 4.5PN order]

4PN: state-of-the-art on equations of motion

$$\frac{\mathrm{d}v_1^i}{\mathrm{d}t} = -\frac{Gm_2}{r_{12}^2}n_{12}^i + \overbrace{\frac{1}{c^2}\left\{\left[\frac{5G^2m_1m_2}{r_{12}^3}+\frac{4G^2m_2^2}{r_{12}^3}+\cdots\right]n_{12}^i+\cdots\right\}}^{\text{1PN Lorentz-Droste-Einstein-Infeld-Hoffmann term}}$$

$$+ \underbrace{\frac{1}{c^4}[\cdots]}_{\text{2PN}} + \underbrace{\frac{1}{c^5}[\cdots]}_{\substack{\text{2.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^6}[\cdots]}_{\text{3PN}} + \underbrace{\frac{1}{c^7}[\cdots]}_{\substack{\text{3.5PN}\\\text{radiation reaction}}} + \underbrace{\frac{1}{c^8}[\cdots]}_{\substack{\text{4PN}\\\text{conservative \& radiation tail}}} + \mathcal{O}\left(\frac{1}{c^9}\right)$$

3PN

[Jaranowski & Schäfer 1999; Damour, Jaranowski & Schäfer 2001ab]	ADM Hamiltonian
[Blanchet-Faye-de Andrade 2000, 2001; Blanchet & Iyer 2002]	Harmonic EOM
[Itoh & Futamase 2003; Itoh 2004]	Surface integral method
[Foffa & Sturani 2011]	Effective field theory

4PN

[Jaranowski & Schäfer 2013; Damour, Jaranowski & Schäfer 2014, 2015]	ADM Hamiltonian
[Bernard, Blanchet, Bohé, Faye, Marchand & Marsat 2015, 2016, 2017ab]	Fokker Lagrangian
[Foffa & Sturani 2012, 2013] (partial results)	Effective field theory

The Fokker Lagrangian approach to the 4PN EOM

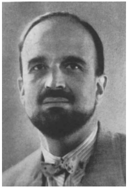
Based on collaborations with

Laura Bernard, **Alejandro Bohé**, **Guillaume Faye**,
Tanguy Marchand & **Sylvain Marsat**

[PRD **93**, 084037 (2016); **95**, 044026 (2017); **96**, 104043 (2017); **97**, 044023 (2018); PRD **97**, 044037 (2018)]

Fokker action of N particles [Fokker 1929]



1. Gauge-fixed Einstein-Hilbert action for N point particles

$$S_{\text{g.f.}} = \frac{c^3}{16\pi G}\int \mathrm{d}^4x\,\sqrt{-g}\Big[R \underbrace{-\frac{1}{2}g_{\mu\nu}\Gamma^\mu\Gamma^\nu}_{\text{Gauge-fixing term}}\Big]$$

$$- \sum_A \underbrace{m_A c^2 \int \mathrm{d}t\,\sqrt{-(g_{\mu\nu})_A\, v_A^\mu v_A^\nu / c^2}}_{N \text{ point particles}}$$

2. Fokker action is obtained by inserting an explicit PN solution of the Einstein field equations

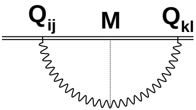
$$g_{\mu\nu}(\mathbf{x},t) \longrightarrow \overline{g}_{\mu\nu}(\mathbf{x};\boldsymbol{y}_B(t),\boldsymbol{v}_B(t),\cdots)$$

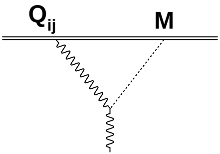
3. The PN equations of motion of the N particles (self-gravitating system) are

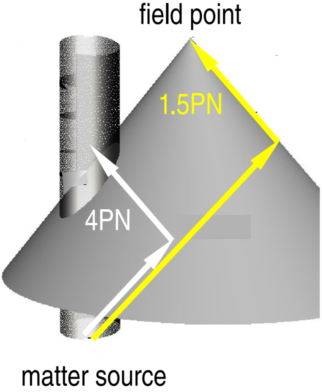
$$\boxed{\frac{\delta S_{\mathsf{F}}}{\delta \boldsymbol{y}_A} \equiv \frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{y}_A} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L_{\mathsf{F}}}{\partial \boldsymbol{v}_A}\right) + \cdots = 0}$$

The gravitational wave tail effect

[Blanchet & Damour 1988; Blanchet 1993, 1997; Foffa & Sturani 2011; Galley, Leibovitz, Porto & Ross 2016]







- In the near zone (4PN effect)

$$S^{\text{tail}} = \frac{G^2 M}{5c^8} \iint \frac{\mathrm{d}t\mathrm{d}t'}{|t-t'|} I^{(3)}_{ij}(t)\, I^{(3)}_{ij}(t')$$

- In the far zone (1.5PN effect)

$$h^{\text{tail}}_{ij} = \frac{4G}{c^4 r}\frac{GM}{c^3}\int_{-\infty}^{t} \mathrm{d}t'\, I^{(4)}_{ij}(t') \ln\left(\frac{t-t'}{\tau_0}\right)$$

Problem of the IR divergences

1. The tail effect implies the appearance of IR divergences in the Fokker action at the 4PN order
2. Our initial calculation of the Fokker action was based on the Hadamard regularization to treat the IR divergences (FP procedure when $B \to 0$)
3. However computing the conserved energy and periastron advance for circular orbits we found it does not agree with GSF calculations
4. The problem was due to the HR and conjectured that a different IR regularization would give (modulo shifts)

$$L = L^{\text{HR}} + \underbrace{\frac{G^4 m\, m_1^2 m_2^2}{c^8 r_{12}^4}\Big(\delta_1 (n_{12}v_{12})^2 + \delta_2 v_{12}^2\Big)}_{\text{two ambiguity parameters } \delta_1 \text{ and } \delta_2}$$

5. Matching with GSF results for the energy and periastron advance uniquely fixes the two ambiguity parameters and we are in complete agreement with the results from the Hamiltonian formalism [DJS]

Dimensional regularization of the IR divergences

- The Hadamard regularization of IR divergences reads

$$I_{\mathcal{R}}^{\mathsf{HR}} = \mathop{\mathrm{FP}}_{B=0} \int_{r>\mathcal{R}} \mathrm{d}^3\mathbf{x} \left(\frac{r}{r_0}\right)^B F(\mathbf{x})$$

- The corresponding dimensional regularization reads

$$I_{\mathcal{R}}^{\mathsf{DR}} = \int_{r>\mathcal{R}} \frac{\mathrm{d}^d\mathbf{x}}{\ell_0^{d-3}} F^{(d)}(\mathbf{x})$$

- The difference between the two regularization is of the type ($\varepsilon = d - 3$)

$$\mathcal{D}I = \sum_q \Bigg[\underbrace{\frac{1}{(q-1)\varepsilon}}_{\text{IR pole}} - \ln\left(\frac{r_0}{\ell_0}\right)\Bigg] \int \mathrm{d}\Omega_{2+\varepsilon}\, \varphi_{3,q}^{(\varepsilon)}(\mathbf{n}) + \mathcal{O}(\varepsilon)$$

Ambiguity-free completion of the 4PN EOM

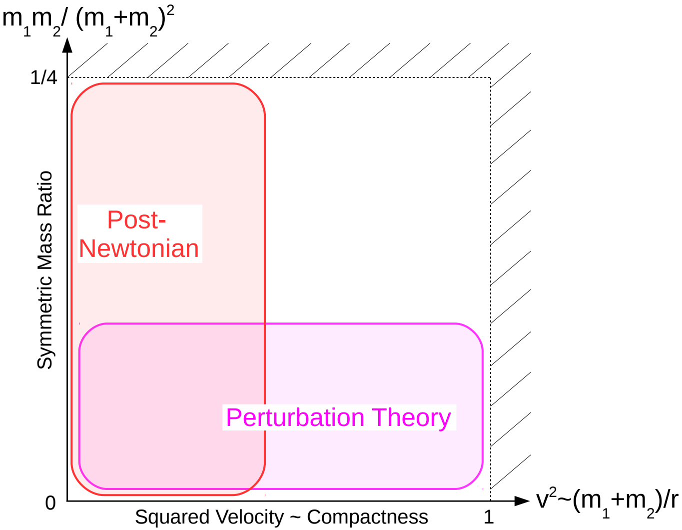
[for more details see the talk by Guillaume Faye in the parallel session BN6]

1. The tail effect contains a UV pole which cancels the IR pole coming from the instantaneous part of the action

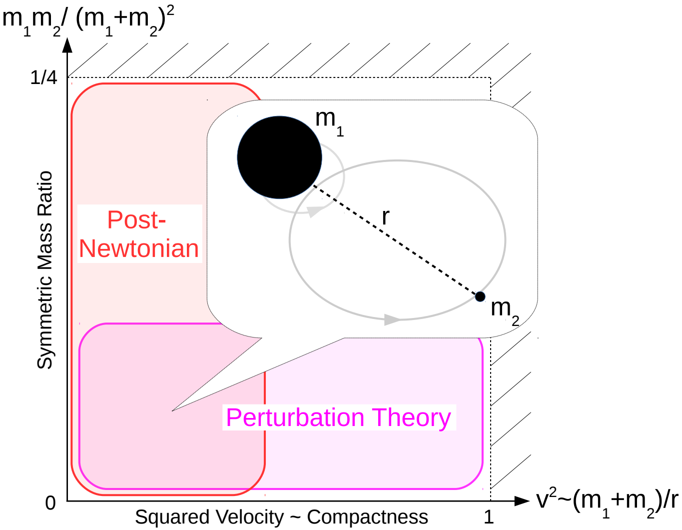
$$g_{00}^{\text{tail}} = -\frac{8G^2M}{5c^8}\,x^{ij}\int_0^{+\infty}\mathrm{d}\tau\left[\ln\left(\frac{c\sqrt{\bar{q}}\,\tau}{2\ell_0}\right)\underbrace{-\frac{1}{2\varepsilon}}_{\text{UV pole}}+\frac{41}{60}\right]I_{ij}^{(7)}(t-\tau)+\mathcal{O}\left(\frac{1}{c^{10}}\right)$$

2. Adding up all contributions we obtain the conjectured form of the ambiguity terms with the correct values of the ambiguity parameters δ_1 and δ_2
3. It is likely that the EFT formalism will also succeed in deriving the full EOM without ambiguities [Porto & Rothstein 2017]
4. The lack of a consistent matching between the near zone and the far zone in the ADM Hamiltonian formalism [DJS] forces this formalism to be still plagued by one ambiguity parameter

Post-Newtonian versus perturbation theory



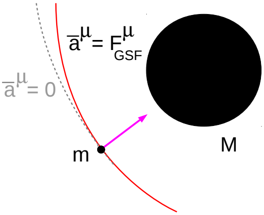
Post-Newtonian versus perturbation theory



Problem of the gravitational self-force (GSF)

[Mino, Sasaki & Tanaka 1997; Quinn & Wald 1997; Detweiler & Whiting 2003]

- A particle is moving on a background space-time of a massive black hole
- Its stress-energy tensor modifies the background gravitational field
- Because of the back-reaction the motion of the particle deviates from a background geodesic hence the gravitational self force

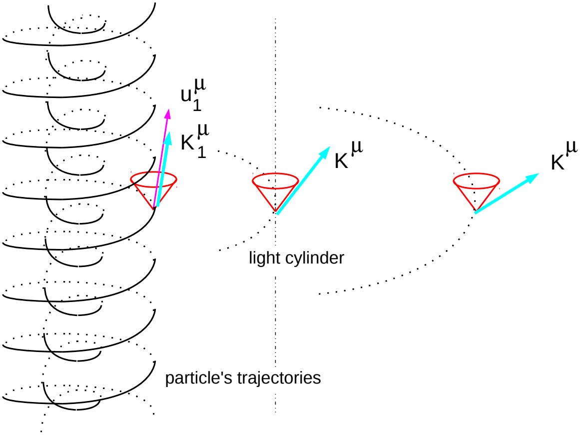


$$\bar{a}^\mu = F^\mu_{\text{GSF}} = \mathcal{O}\left(\frac{m}{M}\right)$$

The GSF is computed to high accuracy by

- numerical methods [Sago, Barack & Detweiler 2008; Shah, Friedmann & Whiting 2014]
- analytical ones [Mano, Susuki & Takasugi 1996ab; Bini & Damour 2013, 2014]

The redshift observable [Detweiler 2008; Barack & Sago 2011]



$$\boxed{K_1^\mu = z_1 u_1^\mu}$$

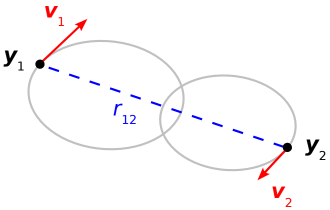
For eccentric orbits one must consider the averaged redshift $\langle z_1 \rangle = \frac{1}{P}\int_0^P \mathrm{d}t\, z_1(t)$

Post-Newtonian calculation of the redshift factor

[Blanchet, Detweiler, Le Tiec & Whiting 2010, 2011; Blanchet, Faye & Whiting 2014, 2015]

In a coordinate system such that $K^\mu \partial_\mu = \partial_t + \omega\, \partial_\varphi$ we have

$$z_1 = \frac{1}{u_1^t} = \left(- \underbrace{(g_{\mu\nu})_1}_{\text{regularized metric}} \frac{v_1^\mu v_1^\nu}{c^2} \right)^{1/2}$$



One needs a self-field regularization

- Hadamard "partie finie" regularization is extremely useful in practical calculations but yields (UV and IR) ambiguity parameters at high PN orders
- Dimensional regularization is an extremely powerful regularization which seems to be free of ambiguities at any PN order

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\text{SF}} = & -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
& + \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_\text{E} - \frac{64}{5}\ln(16y)\right) y^5 \\
& - \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
& + \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
& - \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
& + \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
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&- \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
&+ \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
&- \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
&+ \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

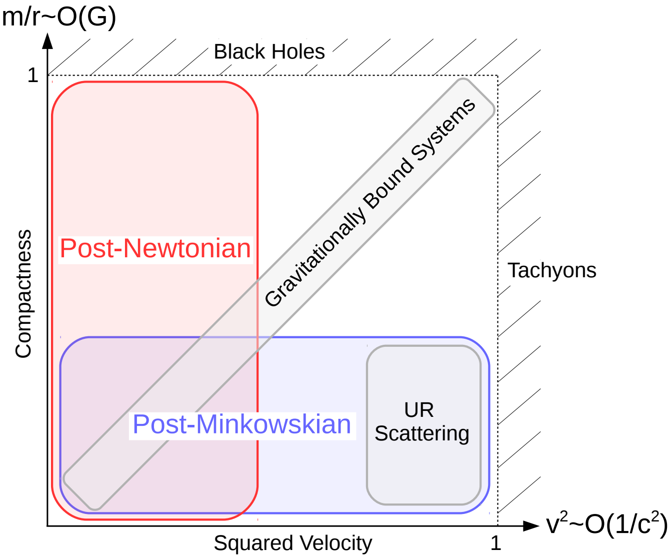
1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the non-linear tails (and tail-of-tails)

Standard PN theory agrees with GSF calculations

$$
\begin{aligned}
u^t_{\mathrm{SF}} = {} & -y - 2y^2 - 5y^3 + \left(-\frac{121}{3} + \frac{41}{32}\pi^2\right) y^4 \\
& + \left(-\frac{1157}{15} + \frac{677}{512}\pi^2 - \frac{128}{5}\gamma_\mathsf{E} - \frac{64}{5}\ln(16y)\right) y^5 \\
& - \frac{956}{105} y^6 \ln y - \frac{13696\pi}{525} y^{13/2} - \frac{51256}{567} y^7 \ln y + \frac{81077\pi}{3675} y^{15/2} \\
& + \frac{27392}{525} y^8 \ln^2 y + \frac{82561159\pi}{467775} y^{17/2} - \frac{27016}{2205} y^9 \ln^2 y \\
& - \frac{11723776\pi}{55125} y^{19/2} \ln y - \frac{4027582708}{9823275} y^{10} \ln^2 y \\
& + \frac{99186502\pi}{1157625} y^{21/2} \ln y + \frac{23447552}{165375} y^{11} \ln^3 y + \cdots
\end{aligned}
$$

1. Integral PN terms such as 3PN permit checking dimensional regularization
2. Half-integral PN terms starting at 5.5PN order permit checking the machinery of non-linear tails (and tail-of-tails)

Post-Newtonian versus post-Minkowskian



The post-Minkowskian approximation

[see *e.g.* Bertotti 1956; Bertotti & Plebanski 1960; Damour & Esposito-Farèse 1996]

- Appropriate for weakly gravitating isolated matter sources $\gamma_{\text{PM}} = \frac{Gm}{c^2 r} \ll 1$

$$\sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \overbrace{\sum_{n=1}^{+\infty} G^n \, h^{\mu\nu}_{(n)}}^{G \text{ labels the PM expansion}}$$

$$\Box h^{\mu\nu}_{(n)} = \frac{16\pi G}{c^4}|g|T^{\mu\nu}_{(n)} + \overbrace{\Lambda^{\mu\nu}_{(n)}[h_{(1)}, \cdots, h_{(n-1)}]}^{\text{know from previous iterations}}$$

- The ultra relativistic gravitational scattering of two particles has been solved up to the 2PM order [Westpfahl *et al.* 1980, 1985; Portilla 1980]
- A closed-form expression for the Hamiltonian of N particles at the 1PM order has been found [Ledvinka, Schäfer & Bičak 2008]
- A renewed interest on the PM approximation and its relation to the PN can be found in the recent literature [see the talk by Donato Bini in the parallel session BN9]

Comparing 4PN with the 1PM approximation

[Blanchet & Fokas 2018]

1. The 1PM field equations of N particles in harmonic coordinates read

$$\Box h^{\mu\nu} = \frac{16\pi}{c^2} \sum_{a=1}^{N} G m_a \int_{-\infty}^{+\infty} \mathrm{d}\tau_a \, u_a^\mu u_a^\nu \delta^{(4)}(x - y_a)$$

2. The Lienard-Wiechert solution is

$$h^{\mu\nu}(x) = -\frac{4}{c^2} \sum_a \frac{G m_a \, u_a^\mu u_a^\nu}{r_a^{\text{ret}} \, (ku)_a^{\text{ret}}}$$

where $r_a^{\text{ret}} = |\boldsymbol{x} - \boldsymbol{x}_a^{\text{ret}}|$ and $(ku)_a^{\text{ret}}$ is the redshift factor

3. In small 1PM terms trajectories are straight lines hence the retardations can be explicitly performed

$$h^{\mu\nu}(\boldsymbol{x}, t) = -\frac{4}{c^2} \sum_a \frac{G m_a \, u_a^\mu u_a^\nu}{r_a \sqrt{1 + (n_a u_a)^2}}$$

Comparing 4PN with the 1PM approximation

[Blanchet & Fokas 2018]

1. This yields the 1PM equations of motion but in PN like form[1]

$$\frac{\mathrm{d}\boldsymbol{v}_a}{\mathrm{d}t} = -\gamma_a^{-2} \sum_{b\neq a} \frac{G m_b}{r_{ab}^2\, y_{ab}^{3/2}} \Big[(2\epsilon_{ab}^2 - 1)\boldsymbol{n}_{ab} + \gamma_b\Big(-4\epsilon_{ab}\gamma_a(n_{ab}v_a) + (2\epsilon_{ab}^2+1)\gamma_b(n_{ab}v_b)\Big)\frac{\boldsymbol{v}_{ab}}{c^2}\Big]$$

2. These equations of motion are conservative and admit a conserved energy

$$E = \sum_a m_a c^2 \gamma_a + \sum_a \sum_{b\neq a} \frac{G m_a m_b}{r_{ab}\, y_{ab}^{1/2}} \Bigg\{ \gamma_a \Big(2\epsilon_{ab}^2 + 1 - 4\frac{\gamma_b}{\gamma_a}\epsilon_{ab}\Big) + \frac{\gamma_b^2}{\gamma_a}(2\epsilon_{ab}^2 - 1)\frac{\dot{r}_{ab}(n_{ab}v_b) - (v_{ab}v_b)}{\big(v_{ab}^2 - \dot{r}_{ab}^2\big)y_{ab} + \frac{\gamma_b^2}{c^2}\big(\dot{r}_{ab}(n_{ab}v_b) - (v_{ab}v_b)\big)^2} \Bigg\}$$

[1] $y_{ab} = 1 + (n_{ab}u_a)^2$ and $\epsilon_{ab} = -(u_a u_b)$

Comparing 4PN with the 1PM approximation

[Blanchet & Fokas 2018]

1. The 1PM Lagrangian in harmonic coordinates is a generalized one

$$L = \sum_a -\frac{m_a c^2}{\gamma_a} + \lambda + \underbrace{\sum_a q_a^i a_a^i}_{\text{accelerations}}$$

2. The 1PM Lagrangian can be computed up to any PN order from the terms of order G in the conserved energy say $E = \sum_a m_a c^2 \gamma_a + \varepsilon$

$$\boxed{\lambda = \text{FP} \int_c^{+\infty} \frac{\mathrm{d}c'}{c}\, \varepsilon\Big(\boldsymbol{x}_a, \frac{\boldsymbol{v}_a}{c'}\Big)}$$

3. We checked in a particular case that the Hamiltonian differs by a canonical transformation from the closed-form expression of the 1PM Hamiltonian in ADM coordinates [Ledvinka, Schäfer & Bičak 2008]
4. All the results reproduce the terms linear in G in the 4PN harmonic coordinates equations of motion and Lagrangian [BBBFMM]