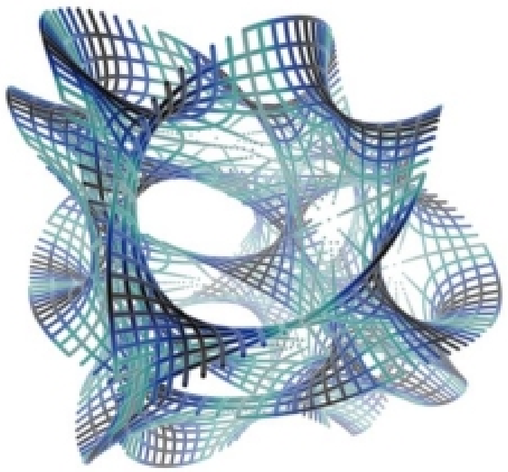
Generalized Geometry and String Theory



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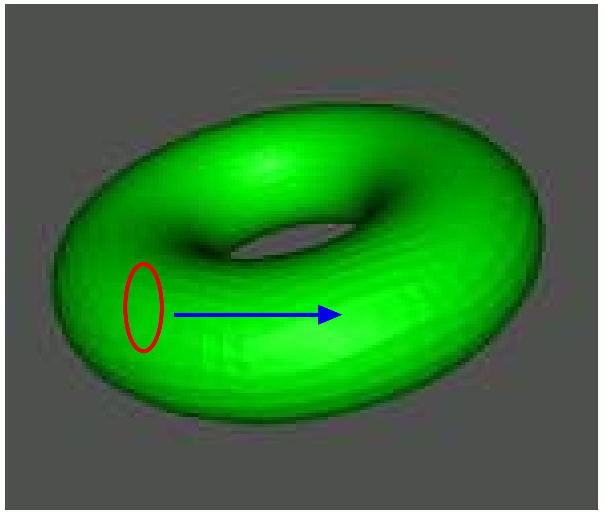


Generalised Geometry

- Much recent work in string theory, supergravity... GG and its generalisations
- Geometric framework for metric + p-form gauge fields
- Unification of various geometric structures
- Organizing principle : Duality symmetries
- New mathematics for old geometries
- New 'non-geometries' in string theory

Generalised Geometry

Studies structures on a d-dimensional manifold M on which there is a natural action of O(d,d) Hitchin

Manifold with metric + B-field, natural action of O(d,d), tangent space "doubled" to $T \oplus T^*$

Extended/Exceptional Geometry: O(d,d) replaced by E_{d+1} acting on extended tangent bundle

Organizing principle : Duality symmetries e.g. O(d,d), E_d

Geometry of String Theory

Spacetime: supergravity background. 10-dimensional manifold M with metric + p-form gauge fields

Type II strings

Metric, signature 9+1 $g_{\mu\nu}$

Dilaton: scalar field ϕ

2-form Gauge field $B_{\mu\nu}$ $H = dB$

p-form gauge fields C_p

$$\delta C_p = d\lambda_{p-1} + \dots, \quad G_{p+1} = G_p + \dots, \quad G_{p+1} = *G_{9-p}$$

Type IIA: p odd. Type IIB: p even

Calabi-Yau

$$M = \mathbb{R}^{3,1} \times K_6$$

Supersymmetric background: admits Killing spinors

Spinor fields generating supersymmetries preserving soln

- 6-fold K: Calabi-Yau if Kahler Ricci-flat
- complex structure J: integrable map $J : TK \to TK$
  $J^2 = -1$
- Hermitian metric g
- Kahler 2-form ω $\qquad \omega_{ij} = g_{ik} J^k{}_j = -\omega_{ji}$
- Holomorphic (3,0) form Ω

$$d\omega = 0, \quad \nabla\omega = 0$$
$$d\Omega = 0, \quad \nabla\Omega = 0$$

Forms from Killing spinor bilinears

Generalisation to FLUX BACKGROUNDS? $\qquad B_2, C_p \neq 0$

Duality Symmetries fromTorus Reductions

- Compactify gravity or supergravity on T^d
- Scalars in coset space G/H,
- Truncated theory: non-comamapct G symmetry
- Full theory: discrete subgroup G(Z)

- General Relativity in 3+1 dimensions on S^1
  EHLERS SYMMETRY $SL(2,\mathbb{R})$
  Scalars in coset space $$\frac{SL(2,\mathbb{R})}{U(1)}$$
- GR in D dimensions, reduced on torus T^d,
  Symmetry $GL(d,\mathbb{R})$
  Scalars g_{ij} in coset space $$\frac{GL(d,\mathbb{R})}{O(d)}$$
- Full theory: Kaluza-Klein spectrum, discrete momentum p, breaking $GL(d,\mathbb{R})$ to $GL(d,\mathbb{Z})$
  Group of large diffeomorphisms of d-torus

- Metric, 2-form gauge field B, reduced on T^d

  Symmetry $O(d,d)$

  Scalars g_{ij} B_{ij} in target space

$$\frac{O(d,d)}{O(d)\times O(d)}$$

- Type II Supergravity, reduced on T^d

  Symmetry E_{d+1} (Cremmer-Julia)

  Scalars $g_{ij}, B_{ij}, C_{i_1\ldots i_p}, \phi$ in target space

$$\frac{E_{d+1}}{H_{d+1}}$$

Full theory: $GL(d,\mathbb{R}) \subseteq G$ breaks to $GL(d,\mathbb{Z})$

String Theory: Discrete Symmetries of full theory

$O(d,d) \to O(d,d;\mathbb{Z})$ <u>T-duality</u>, on p + string winding

$E_{d+1} \to E_{d+1}(\mathbb{Z})$ <u>U-duality</u>, on p + brane wrapping

Generalised Geometry

Conventional manifold M, doubled tangent space

$GL(d,\mathbb{R})$ acting on TM $\Longrightarrow$ $O(d,d)$ acting on $T \oplus T^*$

Natural structure (M,g,B) instead of (M,g)

Doubled Geometry

$GL(d,\mathbb{Z})$ acting on T^d $\Longrightarrow$ $O(d,d,\mathbb{Z})$ acting on T^{2d}

Doubled manifold

Generalised Geometry

Hitchin, Gualtieri

Generalised Vector =Vector + 1-form

$$V = v + \xi \in T \oplus T^* \qquad V^I = \begin{pmatrix} v^i \\ \xi_i \end{pmatrix}$$

O(d,d) $\quad V \to gV$

O(d,d) Metric $\quad \eta = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad g^t \eta g = \eta$

$$\eta(v+\xi, v+\xi) = V^t \eta V = 2 v^i \xi_i$$

Lie Bracket $\Longrightarrow$ Courant Bracket

Generalised Metric $\mathcal{H}_{IJ}$

Positive definite metric on $T \oplus T^*$ compatible with η

$$\eta^{-1}\mathcal{H}\eta^{-1} = \mathcal{H}^{-1}$$

$S = \eta^{-1}\mathcal{H}$ satisfies $S^2 = \mathbb{1}$ Real structure

Parameterised by $G = G^t,\ B = -B^t$ $E = G + B$

Metric G and B-field B

$$\mathcal{H} = \begin{pmatrix} G - BG^{-1}B & BG^{-1} \\ -G^{-1}B & G^{-1} \end{pmatrix}$$

O(d,d):

$$\mathcal{H} \to g^t \mathcal{H} g \qquad E \to (aE + b)(cE + d)^{-1}$$

Parameterise coset $\dfrac{O(d,d)}{O(d) \times O(d)}$

G,B combined in gen metric

Forms on M $\Lambda^\bullet T^*$ $\Longleftrightarrow$ O(d,d) spinors

$$\phi = \alpha + \beta_i dX^i + \gamma_{ij} dX^i \wedge dX^j + \delta_{ijk} dX^i \wedge dX^j \wedge dX^k + \ldots$$

2^d components

Clifford action on forms

$$V = v + \xi \in T \oplus T^*$$

$$\Gamma_V : \Lambda^\bullet T^* \to \Lambda^\bullet T^*$$

$$\Gamma_V : \phi \mapsto \iota_v \phi + \xi \wedge \phi$$

$$\Gamma_V \Gamma_{V'} + \Gamma_{V'} \Gamma_V = -2\eta(V, V')\mathbb{1}$$

Even forms

$$C_0 + C_2 + C_4 + \ldots$$

Chiral spinor

$$C^+$$

Odd forms

$$C_1 + C_3 + C_5 + \ldots$$

Anti-Chiral spinor

$$C^-$$

Generalised Complex Structure $\mathcal{J}$ Hitchin

Endomorphism of $T \oplus T^*$ $\mathcal{J}^2 = -\mathbb{1}$

Courant-integrable η hermitian

Interpolates between symplectic & complex structures, generalising both.

Generalised Kahler Gualtieri

$\mathcal{H}, \mathcal{J}$ $S = \eta^{-1}\mathcal{H}$

$[S, \mathcal{J}] = 0$ $\mathcal{J}' = \mathcal{J} S$

Generalised Kahler $\longleftrightarrow$ **Bihermitian Geometry**

Gualtieri

$$(M;\mathcal{H},\mathcal{J},\mathcal{J}') \qquad\qquad (M;G,B,J_+,J_-)$$

$$\mathcal{H}_{IJ} \qquad\qquad G_{ij},B_{ij}$$

$$\mathcal{J}^I{}_J \qquad\qquad (J_\pm)^i{}_j$$

Two Complex structures on M, G Hermitian

$$\nabla^\pm J_\pm = 0 \qquad \nabla^\pm = \nabla \pm G^{-1}H$$

(2,2) SUSY Sigma Model with torsion Gates,CMH,Rocek

SUSY analysis gives general structure, potential

Lindstrom, Rocek, von Unge, Zabzine

Generalised Calabi-Yau

- Gen Complex geometry can be encoded by pair of (pure) O(d,d) spinors i.e. differential forms $\Phi \sim \Omega, \Psi \sim e^{B+i\omega}$
- <u>Generalised CY</u>: these forms closed, and the spinors non-degenerate Hitchin, Gualtieri
- RR fields are also an O(d,d) spinor
- <u>Gen CY with flux</u>: Conditions for SUSY have elegant formulation in terms of these spinors Grana, Minasian, Petrini, Tomasiello

Strings on Circle

$$M = S^1 \times X$$

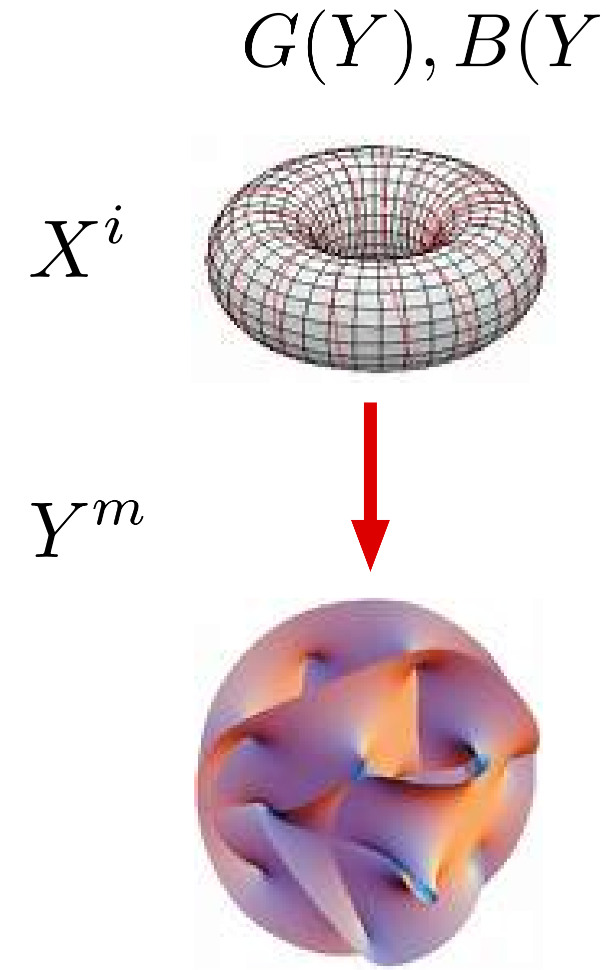
Discrete momentum p=n/R
If it winds m times round S^1, winding energy w=mRT
Energy = p^2+w^2+....

<u>T-duality</u>: Symmetry of string theory

$$\begin{array}{ccc} p & \leftrightarrow & w \\ m & \leftrightarrow & n \\ R & \leftrightarrow & 1/RT \end{array}$$

- Fourier transf of discrete p,w gives periodic coordinates
  $X, \tilde{X}$ Circle + dual circle
- Stringy symmetry, not in field theory
- On d torus, T-duality group $O(d,d;\mathbb{Z})$

T-Duality



- Space has d-torus fibration
- G,B on fibres
- T-Duality O(d,d;Z), mixes G,B
- Mixes Momentum and Winding
- Changes geometry and topology

$$E \to (aE+b)(cE+d)^{-1}$$

$$h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(d,d;Z) \qquad E_{ij} = G_{ij} + B_{ij}$$

On circle, radius R: $\quad O(1,1;\mathbb{Z}) = \mathbb{Z}_2 : R \mapsto \dfrac{1}{R}$

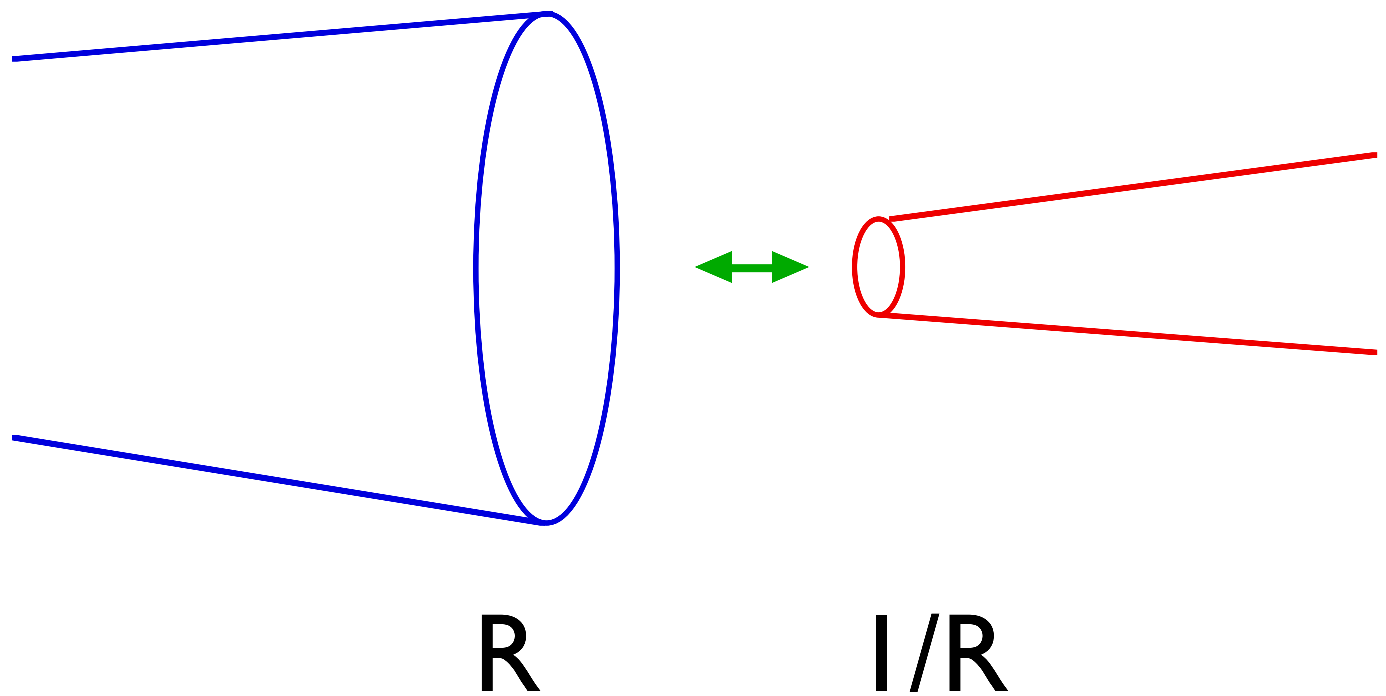
Symmetry & Geometry

- Spacetime constructed from local patches
- All symmetries of physics used in patching
- Patching with diffeomorphisms, gives manifold
- Patching with gauge symmetries: bundles
- String theory has new symmetries, not present in field theory. New non-geometric string backgrounds

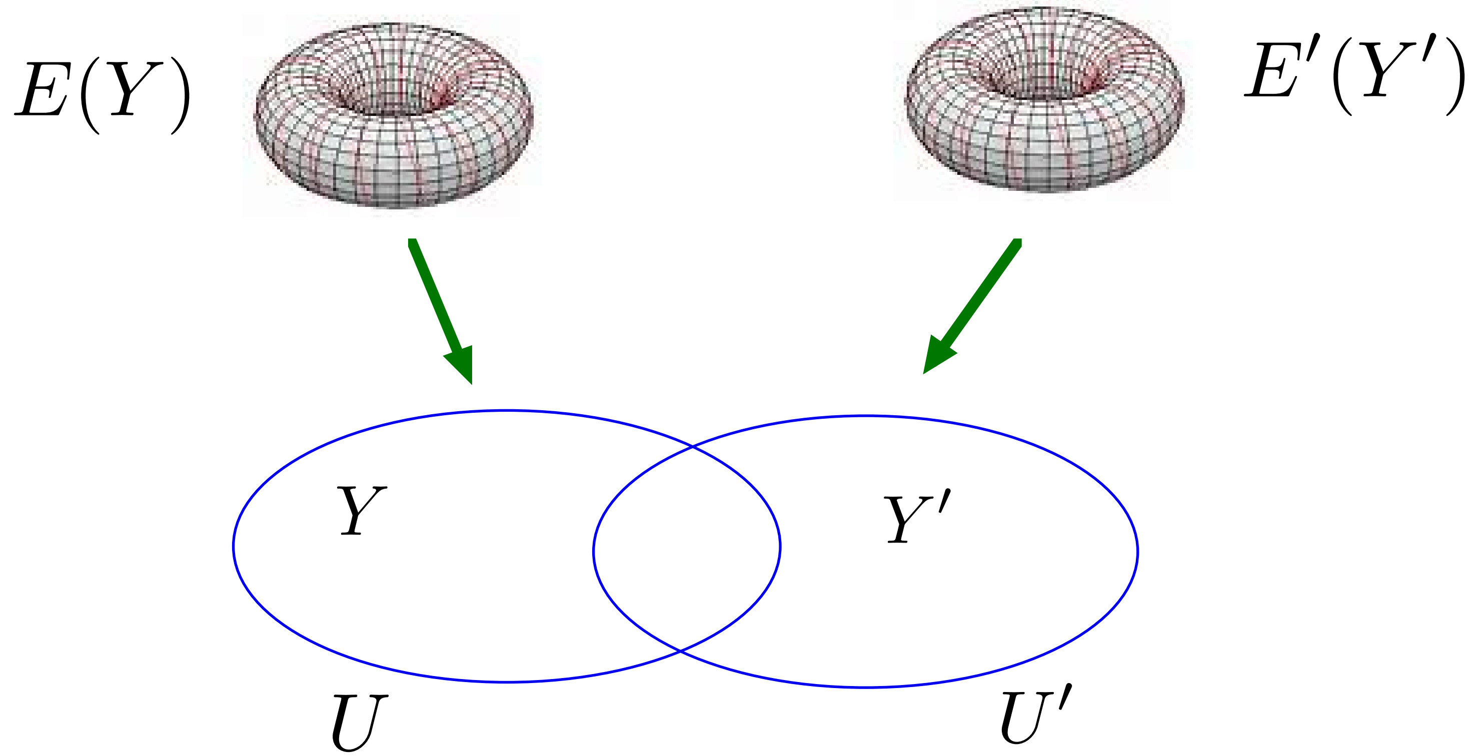
Hull

- Patching with T-duality: **T-FOLDS**
- Patching with U-duality: **U-FOLDS**

T-fold patching



Glue big circle (R) to small (1/R)
Glue momentum modes to winding modes
(or linear combination of momentum and winding)
Not conventional smooth geometry



Torus fibration

Geometric background: G, H=dB tensorial

T-fold: Transition functions involve T-dualities (as well as diffeomorphisms and 2-form gauge transformations)

E=G+B Non-tensorial

$O(d,d;\mathbb{Z})$ $\qquad E' = (aE+b)(cE+d)^{-1}$ in $U \cap U'$

Glue using T-dualities also → **T-fold**

Physics smooth, as T-duality a symmetry

Not conventional smooth geometry

Fixes moduli!

Double Field Theory

Hull & Zwiebach

- From sector of String Theory. Features some stringy physics, including T-duality, in simpler setting
- Strings see a doubled space-time
- Necessary consequence of string theory
- Needed for non-geometric backgrounds

Strings on a Torus

- States: momentum p, winding w
- String: Infinite set of fields $\psi(p, w)$
- Fourier transform to doubled space: $\psi(x, \tilde{x})$
- "Double Field Theory" from closed string field theory. Some non-locality in doubled space
- Subsector? e.g. $g_{ij}(x, \tilde{x}),\ b_{ij}(x, \tilde{x}),\ \phi(x, \tilde{x})$

Double Field Theory

- Double field theory on doubled torus
- General solution of string theory: involves doubled fields $\psi(x, \tilde{x})$
- *Real* dependence on *full* doubled geometry, dual dimensions not auxiliary or gauge artifact. Double geom. *physical* and *dynamical*
- *Strong constraint* restricts to subsector in which extra coordinates auxiliary: get conventional field theory locally. Recover Siegel's duality covariant formulation of (super)gravity, GG.

O(D,D) covariant action using generalised metric, dilaton d

$$S = \int dx d\tilde{x}\, e^{-2d}\, L$$

$$L = \frac{1}{8}\mathcal{H}^{MN}\partial_M \mathcal{H}^{KL}\,\partial_N \mathcal{H}_{KL} - \frac{1}{2}\mathcal{H}^{MN}\partial_N \mathcal{H}^{KL}\,\partial_L \mathcal{H}_{MK}$$
$$- 2\,\partial_M d\,\partial_N \mathcal{H}^{MN} + 4\mathcal{H}^{MN}\,\partial_M d\,\partial_N d$$

Hohm, Hull & Zwiebach

L cubic! Indices raised and lowered with η

If independent of dual coord, $\tilde{X}$
Gives usual action (+ surface term)

$$\int dx \sqrt{-g} e^{-2\phi}\left[R + 4(\partial\phi)^2 - \frac{1}{12}H^2\right]$$

Extended Geometry, M-Theory

$T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$

Hull; Pacheco & Waldram

- Extended geometry: extends tangent space
- Metric and 3-form gauge field, action of exceptional U-duality group
- 10-d type II and 11-d sugra in terms of extended geometry

Coimbra, Strickland-Constable & Waldram

- Double field theory generalises

Berman, Perry et al, Hohm & Sambtleben, ...

Generalised Parallelalisability

Lee, Strickland Constable, Waldram

- Generalised (or extended) tangent space is generalised parallelalisable if it admits a global frame. Important examples include S^3,S^4,S^7
- Explains consistent truncations of supergravity on S^3,S^4,S^7
- Gives insights into exotic supergravity gaugings of Dell'Agata, Inverso, Trigiante

Conclusions

- Generalised geometry gives elegant formulation for gravity + p-form gauge fields
- Unified treatment of various structures, allows general results
- Generalised complex geometry, generalised CY, superymmetric spaces. Gen parallelisability.
- String theory sees a doubled or extended spacetime, physical extra dimensions
- String theory has non-geometric solns, which look geometric in doubled formulation

Doubled Geometry for T-fold

Hull

T^d torus fibres have doubled coords $\mathbb{X}^I = \begin{pmatrix} X^i \\ \widetilde{X}_i \end{pmatrix}$ $I = 1, ..., 2d$

Transforms linearly under $O(d,d;\mathbb{Z})$
T-fold transition: mixes $X, \tilde{X}$
No global way of separating "real" space coordinate X from "auxiliary" $\widetilde{X}$

Duality covariant formulation in terms of $\mathbb{X}$
Transition functions $O(d,d;\mathbb{Z}) \subset GL(2d;\mathbb{Z})$
can be used to construct bundle with fibres T^{2d}

Doubled space is smooth manifold!

Sigma Model on doubled space

Type I Extended Geometry

Action of O(d,d)

$T \oplus T^*$

cf Brane charges

Generalised metric

G, B_2 $\Longleftrightarrow$ $\mathcal{H}_{IJ} \in \dfrac{O(d,d)}{O(d) \times O(d)}$

Type I Extended Geometry

Action of O(d,d)

$T \oplus T^*$

cf Brane charges

G, B_2 $\Longleftrightarrow$ Generalised metric

$$\mathcal{H}_{IJ} \in \frac{O(d,d)}{O(d) \times O(d)}$$

Type II: O(d,d) $\longrightarrow$ E_{d+1}, add RR fields

Type I Extended Geometry

Action of O(d,d)

$T \oplus T^*$

cf Brane charges

Generalised metric

G, B_2 $\Longleftrightarrow$ $\mathcal{H}_{IJ} \in \dfrac{O(d,d)}{O(d) \times O(d)}$

Type M: O(d,d) $\Rightarrow$ E_d, $B_2 \Rightarrow C_3$

$T \oplus T^* \Rightarrow T \oplus \Lambda^2 T^*$

Type II Extended Geometry $d \leq 4$

Action of E$_{d+1}$

$T \oplus T^* \oplus S^{\pm}$

cf Brane charges

Generalised metric

$G, B_2, \Phi, C^{\mp}$ $\longleftrightarrow$

$$\mathcal{H}_{IJ} \in \frac{E_{d+1}}{H_{d+1}}$$

IIB C^+ $C_0 + C_2 + C_4 + ...$

RR fields

IIA C^- $C_1 + C_3 + C_5 + ...$

Type II Extended Geometry $\quad d \leq 6$

Action of E_{d+1}

$$T \oplus T^* \oplus S^{\pm} \oplus \Lambda^5 T \oplus \Lambda^5 T^*$$

cf Brane charges

$$G, B_2, \Phi, C^{\mp}, \tilde{B}_6 \quad \Longleftrightarrow$$

Generalised metric

$$\mathcal{H}_{IJ} \in \frac{E_{d+1}}{H_{d+1}}$$

IIB $\quad C^+ \quad C_0 + C_2 + C_4 + \ldots$

IIA $\quad C^- \quad C_1 + C_3 + C_5 + \ldots$

RR fields

Type M Extended Geometry $d \leq 4$

Action of E$_d$

$T \oplus \Lambda^2 T^*$

cf Brane charges

Generalised metric

G, C_3 $\Longleftrightarrow$ $\mathcal{H}_{IJ} \in \dfrac{E_d}{H_d}$

Type M Extended Geometry $d \leq 7$

Action of E_d

cf Brane charges

$$T \oplus \Lambda^2 T^* \oplus \Lambda^5 T^* \oplus \Lambda^6 T$$

Generalised metric

$$G, C_3, \tilde{C}_6 \quad \Longleftrightarrow \quad \mathcal{H}_{IJ} \in \frac{E_d}{H_d}$$

- More general non-geometric backgrounds. Gives uplift of GENERIC gauged Sugras

Shelton, Taylor & Wecht 2005

Dabholkar & Hull 2005

- Explicit doubled geometries constructed for T-folds and "spaces with R-flux"

Hull & Reid-Edwards 2008-9

- Sigma models on doubled spaces; quantisation

Hull 2004-6

- Other approaches to quantisation

Tseytlin; Berman, Thompson, Copland; Hackett-Jones & Motsopoulos

Strings on T^d

$$X = X_L(\sigma+\tau) + X_R(\sigma-\tau), \qquad \tilde{X} = X_L - X_R$$

X conjugate to momentum, $\tilde{X}$ to winding no.

$$dX = *d\tilde{X} \qquad\qquad \partial_a X = \epsilon_{ab}\partial^b \tilde{X}$$

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Need "auxiliary" $\tilde{X}$ for interacting theory

i) Vertex operators $e^{ik_L\cdot X_L}, \quad e^{ik_R\cdot X_R}$

ii) String field Kugo & Zwiebach $\Phi[x,\tilde{x},a,\tilde{a}]$

Strings on T^d

$$X = X_L(\sigma+\tau) + X_R(\sigma-\tau), \qquad \tilde{X} = X_L - X_R$$

X conjugate to momentum, $\tilde{X}$ to winding no.

$$dX = *d\tilde{X} \qquad\qquad \partial_a X = \epsilon_{ab}\partial^b \tilde{X}$$

Strings on torus see **DOUBLED GEOMETRY**!

T-duality group $O(d,d;\mathbb{Z})$

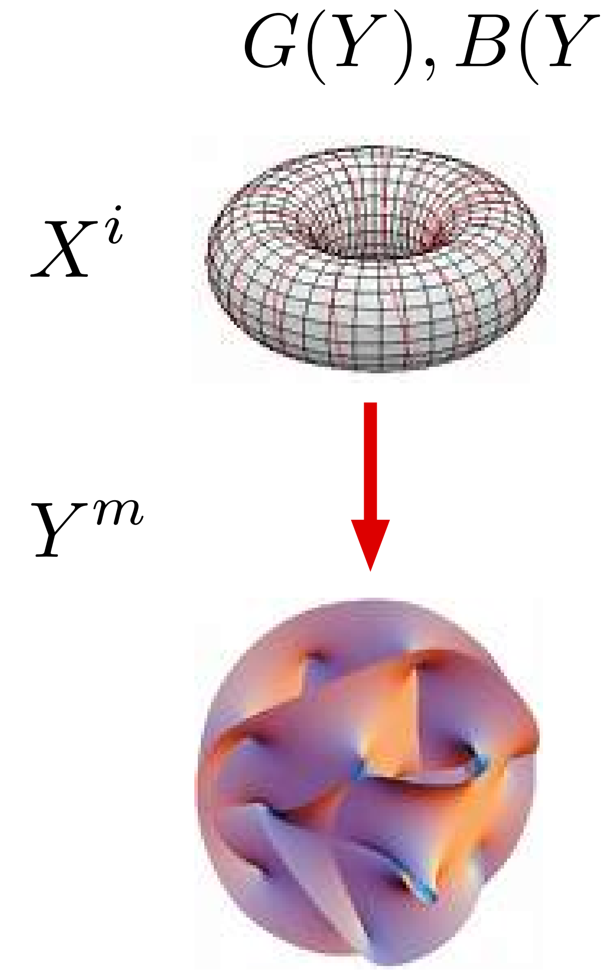
Doubled Torus 2d coordinates

Transform linearly under $O(d,d;\mathbb{Z})$

$$X \equiv \begin{pmatrix} \tilde{x}_i \\ x^i \end{pmatrix}$$

Sigma model on doubled torus Tseytlin; Hull

T-Duality



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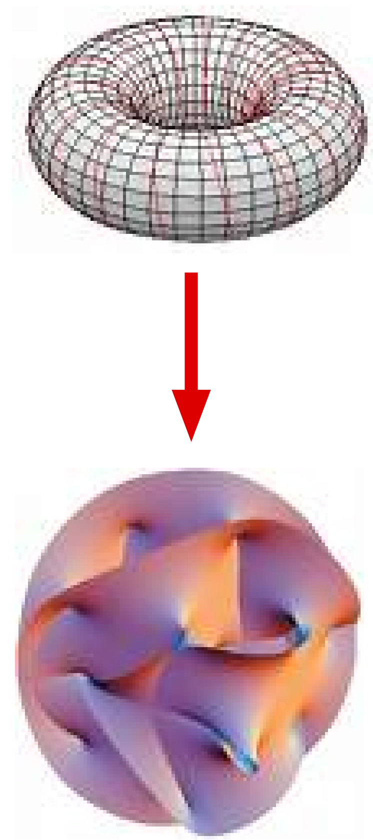
On circle, radius R: $\quad O(1,1;\mathbb{Z}) = \mathbb{Z}_2 : R \mapsto \dfrac{1}{R}$

T-Duality



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