

About the Statistical Properties of Cosmological Billiards

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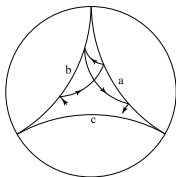
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Summary

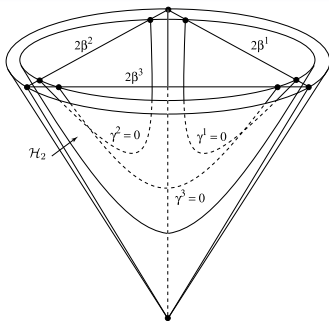
- Introduction: historical viewpoint
- Descriptions of billiards
- Integral invariants
- Parametrization of the dynamics
- Epochs and eras
- Maps
- Recovering information from the maps
- Applications
- Outlook

Historical Viewpoint

- 1969 BKL: chaotic behaviour of generic cosmological solution locally described by ODE's
⇒ approximate solution of ODE's
⇒ statistical analysis by BKL, LLK
- 1969 Misner, 1972 Chitré: billiard on a Lobachevsky plane
- 1983 CB-LKSKS statistical analysis of the map
 $(x_+, x_-) \rightarrow (x'_+, x'_-)$



- 1997 KM: link between billiard and discrete map
- 2001 DH, 2002 DHN Hidden mathematical structure (Kac-Moody) ⇒ general cosmological billiards



Lorentzian space

$$\mathcal{L} = G_{ab} \dot{\beta}^a \dot{\beta}^b - V(\beta)$$

$$V(\beta) = \sum c_A e^{-2w_A(\beta)}$$

$w_A(\beta)$ walls

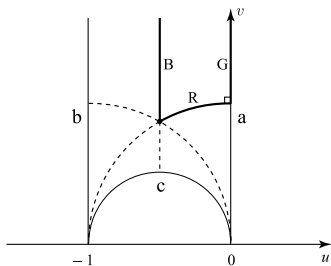
Gravitational walls: $\alpha_{abc} = \beta^a + \beta^b - \beta_c + \sum \beta$

Symmetry walls: $w_{ab}^{sym} = \beta^b - \beta^a$

⇒ various types of billiards

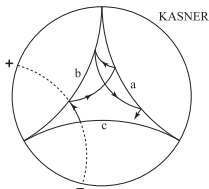
- big billiard
- small billiard

⇒ projecting Lorentzian (β) space on Lobachevsky plane



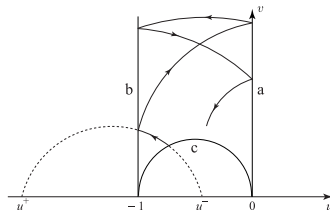
$$\gamma^2 = -1$$
$$\beta^a = \rho \gamma^a$$

Two complementary descriptions of billiards



Unitary disk

more symmetric
symmetry group: order 6



Poincaré half plane

less symmetric
technically useful

In the Poincaré half plane

- each collision is described by

$$z' = -\frac{a\bar{z} + b}{c\bar{z} + d},$$
$$ad - bc = 1.$$

- endpoints parametrized by u^+ s.t.

$$p_1^{BKL}(u) = -\frac{u_+}{u_+^2 + u_+ + 1},$$
$$p_2^{BKL}(u) = \frac{u_+ + 1}{u_+^2 + u_+ + 1},$$
$$p_3^{BKL}(u) = \frac{u_+(u_+ + 1)}{u_+^2 + u_+ + 1}.$$

Integral invariants for billiards

Poincaré-Cartan 2-form

$$\omega_{PC}^{(2)} := -d\sigma_{PC}^{(1)} = dq^i \wedge dp_i - dt \wedge dH(q, p, t)$$

invariant under both

- “epoch” free evolution and
- “collisions on walls”.

⇒ invariant 2-form from collision to collision.

Can be reduced to $H = E$ energy-shell hypersurface:
phase-space dim 3, where

$$H_\gamma(u, v, p_u, p_v) = \frac{1}{2}v^2 (p_u^2 + p_v^2) + V_\infty(\gamma(u, v))$$

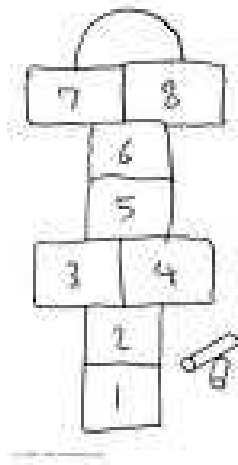
$$\omega_{on}^{reduced} = [du \wedge dp_u + dv \wedge dp_v]_{v=const} = du \wedge dp_u = 2 \frac{du^+ \wedge du^-}{(u^+ - u^-)^2}$$

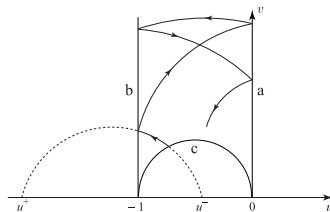
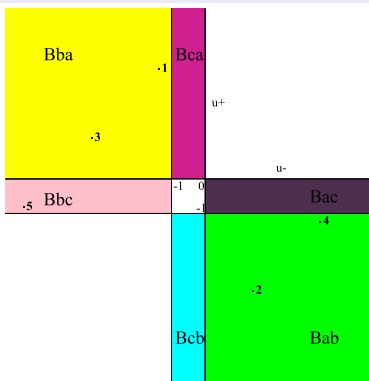
- clearly invariant under free evolution
- also invariant under collisions $u'_\pm = -\frac{au_\pm + b}{cu_\pm + d}$

this invariant 2-form is linked to the Liouville 3-form by

$$\Omega_L^{(3)} = \omega^{(2)} \wedge ds$$

Hopscotch dynamics of the big billiard



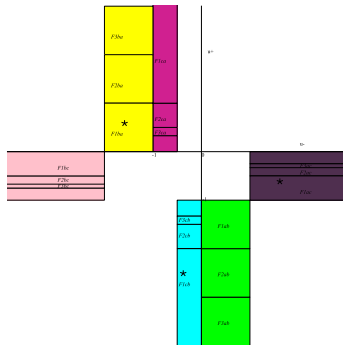
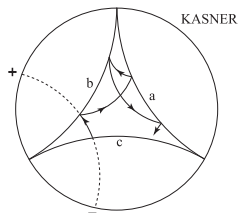


but $\int_{\text{Hopscotch}} \omega^{(2)} = \infty$

delicate “infinite ergodic theory”

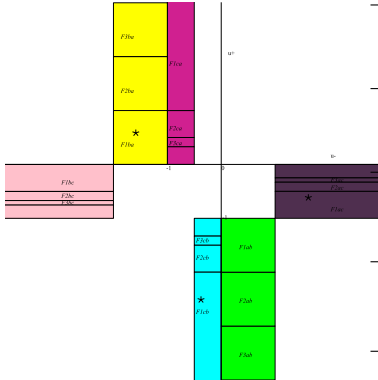
BKL solution: focus on **era** hopscotch instead of epoch hopscotch

first epoch $\rightarrow$ next first epoch



Further investigation

Era hopscotch



F_{ab}	n^{ab} odd n^{ab} even	F'_{bc} F'_{ac}	$u_{F_{bc}} = -u_{F_{ab}} - n^{ab} - 1$ $u_{F_{ac}} = u_{F_{ab}} + n^{ab}$
F_{ba}	n^{ba} odd n^{ba} even	F'_{ac} F'_{bc}	$u_{F_{ac}} = -u_{F_{ba}} + n^{ba} - 1$ $u_{F_{bc}} = u_{F_{ba}} - n^{ba}$
F_{ac}	n^{ac} odd n^{ac} even	F'_{cb} F'_{ab}	$u_{F_{cb}} = -\frac{1}{n^{ac}+1+\frac{1}{u_{F_{ac}}}}$ $u_{F_{ab}} = \frac{1}{n^{ac}+\frac{1}{u_{F_{ac}}}}$
F_{ca}	n^{ca} odd n^{ca} even	F'_{ab} F'_{cb}	$u_{F_{ab}} = \frac{1}{n^{ca}-1-\frac{1}{u_{F_{ca}}}}$ $u_{F_{cb}} = -\frac{1}{n^{ca}-\frac{1}{u_{F_{ca}}}}$
F_{bc}	n^{bc} odd n^{bc} even	F'_{ca} F'_{ba}	$u_{F_{ca}} = -1 + \frac{1}{n^{bc}+1-\frac{1}{1+u_{F_{bc}}}}$ $u_{F_{ba}} = -1 - \frac{1}{n^{bc}-\frac{1}{1+u_{F_{bc}}}}$
F_{cb}	n^{cb} odd n^{cb} even	F'_{ba} F'_{ca}	$u_{F_{ba}} = -1 - \frac{1}{n^{cb}-1+\frac{1}{1+u_{F_{cb}}}}$ $u_{F_{ca}} = -1 + \frac{1}{n^{cb}+\frac{1}{1+u_{F_{cb}}}}$

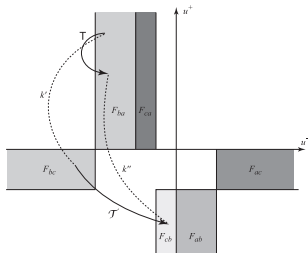
invariant measure $\omega^{(2)} = 2 \frac{du^+ \wedge du^-}{(u^+ - u^-)^2}$

$\Rightarrow$ “probability” for relative frequencies of hopscotch, e.g.

- $\text{prob}(F_{ba}^1) = \text{prob}(F_{ab}^1) = \dots \sim 0.42/6 \sim 0.07$
- $\text{prob}(F_{ba}^2) = \text{prob}(F_{ab}^2) = \dots \sim 0.17/6 \sim 0.03$

$\Rightarrow$ transition probability $\frac{F_{ba \rightarrow ac}}{F_{ba \rightarrow bc}} \sim 1.8$
“anisotropic” behaviour

big billiard map $\mathcal{T}$
 CB-LKSKS map T



one iteration $\mathcal{T} = k_i^{-1}(1)\mathsf{T}k_j(0)$
 N iterations $\mathcal{T}^N = k_{i'}^{-1}(N)\mathsf{T}^N k_j(0)$

Recovering full information about the era hopscotch from T

in F_{ba} ,

$u^+ \in [0, \infty]$, including $[0, 1]$, $u^+ + 1 = [n_1; n_2, n_3, \dots]$

$u^- \in [-2, -1]$, $u^- = -1 - [m_1, m_2, m_3, \dots]$

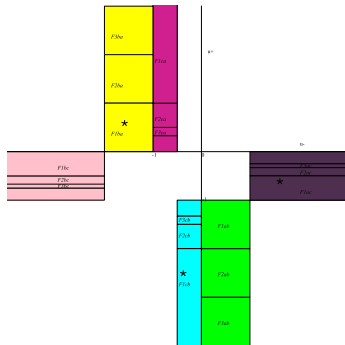
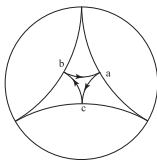
$$\epsilon_i = (-)^{n_i+1} \Rightarrow$$

- rotation $\rho_N = e^{-i\frac{2\pi}{3}(\epsilon_1 + \epsilon_1\epsilon_2 + \dots)}$
- permutation $\eta_N = (-)^{\epsilon_1\epsilon_2\cdots\epsilon_N}$

after N eras, the $(N+1)$ -th era takes place
in the corner defined by ρ_N and
with direction defined by η_N

Periodic big-billiard orbits

Different from usual BKL periodicity



$$u^+ = \frac{\sqrt{5}-1}{2} = [0; 1, 1, 1, ..]$$

 $Tu = u$, but

- BKL periodicity: 1;
- Hopscotch periodicity: 3.

Outlook

It is possible to apply this kind of investigation to other structures, such as

- small billiard in $3 + 1$ dimensions
- higher-dimensional billiards

Thank you!