

Poynting-Robertson effect in black hole spacetimes

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Collaboration

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PR effect

(Poynting 1903, Newtonian gravity;
Robertson 1937, linearized GR).



Consider a small body orbiting a star.

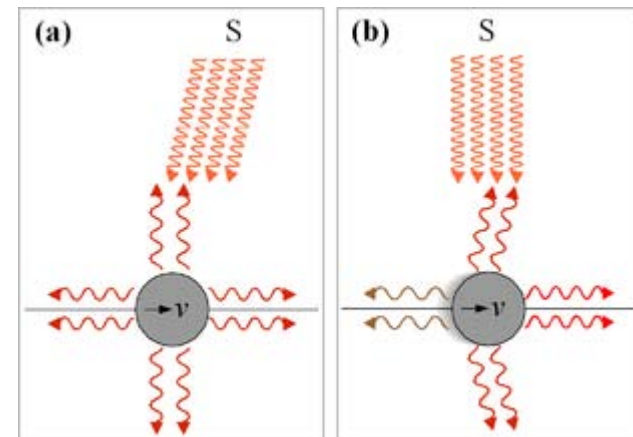
The light emitted by the star exerts a radiation pressure on the body whose direct effect is a **drag force**, which causes the body to fall into the star, unless the body is so small that it is pushed away from the star itself.

Radiation from the Sun (S) and thermal radiation from a particle seen (a) from an observer moving with the particle and (b) from an observer at rest with respect to the Sun.

The effect can be understood in two ways, depending on the reference frame chosen.

From the perspective of the grain of dust circling the Sun, the Sun's radiation appears to be coming from a slightly forward direction (aberration of light). Therefore the absorption of this radiation leads to a force with a component against the direction of movement. (The angle of aberration is extremely small since the radiation is moving at the speed of light while the dust grain is moving many orders of magnitude slower than that.)

From the perspective of the solar system as a whole, the dust grain absorbs sunlight entirely in a radial direction, thus the grain's angular momentum remains unchanged. However, in absorbing photons, the dust acquires added mass via mass-energy equivalence. In order to conserve angular momentum (which is proportional to mass), the dust grain must drop into a lower orbit.



PR effect



The drag force is “naively” understood as an aberration effect: if the body is in a circular orbit, for example, the radiation pressure is radially outward from the star, but in the rest frame of the body, the radiation appears to be coming from a direction slightly towards its own direction of motion, and hence a backwards component of force is exerted on the body which acts as a drag force. If the drag force dominates the outward radial force, the body falls into the star.



For the case in which a body is momentarily at rest, a critical luminosity (similar to the Eddington limit for a star) exists at which the inward gravitational force balances the outward radiation force, a critical value separating radial infall from radial escape.



Similarly for a body initially in a circular orbit, there are two kinds of solutions: those in which the body spirals inward or spirals outward, depending on the strength of the radiation pressure.

Classical understanding of the PR effect

The **PR effect** named after John Henry Poynting and Howard Percy Robertson, is a process by which solar radiation causes a dust grain in the solar system to slowly spiral inward. The drag is essentially a component of radiation pressure tangential to the grain's motion. The first description of this effect, given by Poynting in 1903, was still classically based. Later, in 1937, Robertson described the effect correctly in terms of general relativity.

Note that the *re-emission* of photons, which is isotropic in the frame of the grain, does not affect the dust particle's orbital motion. However, in the frame of the solar system, the emission is beamed anisotropically, and hence the photons carry away angular momentum from the dust grain. It is somewhat counter-intuitive that angular momentum is lost while the orbital motion of the grain is unchanged, but this is an immediate consequence of the dust grain shedding mass during emission and that angular momentum is proportional to mass.

The PR drag can be understood as an effective force (opposite the direction of the dust grain's orbital motion), leading to an increase in the grain's angular momentum. It should be mentioned that while the dust grain thus spirals slowly into the Sun, its orbital speed increases continuously.

The PR force is equal to



$$F_{PR} = \frac{Wv}{c^2} = \frac{r^2}{4c^2} \sqrt{\frac{GM_s L_s^2}{R^5}}$$

where W is the power of the incoming radiation, v is the grain's velocity, c is the speed of light, r the grain's radius, G is the universal gravitational constant, M_s the Sun's mass, L_s is the solar luminosity and R the grain's orbital radius.

Since the gravitational force goes as the cube of the object's radius, R^3 (being a function of its volume) whilst the power it receives and radiates goes as the square of that same radius, R^2 (being a function of its surface), the PR effect is more pronounced for smaller objects. Also, since the Sun's gravity varies $1/R^2$ whereas the PR force varies as $1/R^{2.5}$, the latter gets relatively stronger as the object approaches the Sun, which tends to reduce the eccentricity of the object's orbit in addition to dragging it in.

Rocky dust *particles sized a few micrometers* need a few thousand years to get from 1 AU distance to distances where they evaporate.

For *particles much smaller than this*, radiation pressure, which makes them spiral outwards from the sun, is stronger than the PR effect that makes them spiral inward. For rocky particles about half a μm in diameter, the radiation pressure equals gravity, and they will be always blown out of the solar system even though the PR effect still affects them.

Particles of intermediate size will either spiral inwards or outwards depending on their size and their initial velocity vector.

PR effect and bh spacetimes

- ➡ We consider this problem in the context of a test body in orbit in a spherically symmetric Schwarzschild spacetime without the restriction of slow motion, and then in the larger context of an axially symmetric Kerr spacetime.
- ➡ The finite size of the radiating body is ignored.
- ➡ The photon flux from the central body is modeled by test photons in outward radial motion with respect to the locally nonrotating observers, namely photons with vanishing conserved angular momentum.
- ➡ While developing the basic equations for a stationary axisymmetric spacetime, we have considered their solution only for the Schwarzschild and Kerr spacetimes.

Background
spacetime

GR setting of the problem

Stationary, axisymmetric and reflection-symmetric spacetimes

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{rr}dr^2 + g_{\theta\theta}d\theta^2$$

Metric

$$n = N^{-1}(\partial_t - N^\phi \partial_\phi) \quad N = (-g^{tt})^{-1/2} \quad N^\phi = g_{t\phi}/g_{\phi\phi}$$

Lapse-Shift notation

$$e_{\hat{t}} = n, \quad e_{\hat{r}} = \frac{1}{\sqrt{g_{rr}}} \partial_r, \quad e_{\hat{\theta}} = \frac{1}{\sqrt{g_{\theta\theta}}} \partial_\theta, \quad e_{\hat{\phi}} = \frac{1}{\sqrt{g_{\phi\phi}}} \partial_\phi$$

ZAMOs

$$a(n) = a(n)^{\hat{r}} e_{\hat{r}} + a(n)^{\hat{\theta}} e_{\hat{\theta}} = \partial_{\hat{r}}(\ln N) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln N) e_{\hat{\theta}},$$

$$\theta_{\hat{\phi}}(n) = \theta_{\hat{\phi}}(n)^{\hat{r}} e_{\hat{r}} + \theta_{\hat{\phi}}(n)^{\hat{\theta}} e_{\hat{\theta}} = -\frac{\sqrt{g_{\phi\phi}}}{2N} [\partial_{\hat{r}} N^\phi e_{\hat{r}} + \partial_{\hat{\theta}} N^\phi e_{\hat{\theta}}],$$

$$k_{(\text{lie})}(n) = k_{(\text{lie})}(n)^{\hat{r}} e_{\hat{r}} + k_{(\text{lie})}(n)^{\hat{\theta}} e_{\hat{\theta}} = -[\partial_{\hat{r}}(\ln \sqrt{g_{\phi\phi}}) e_{\hat{r}} + \partial_{\hat{\theta}}(\ln \sqrt{g_{\phi\phi}}) e_{\hat{\theta}}]$$

$$T^{\alpha\beta} = \Phi^2 k^\alpha k^\beta, \quad k^\alpha k_\alpha = 0$$

Photon (test)
field

*Photon field, superposed as a
test field to the gravitational
background*

where k is assumed to be tangent to an affinely parametrized outgoing null geodesic in the equatorial plane, i.e., $k^\alpha \nabla_\alpha k^\beta = 0$ with $k^\theta = 0$. We will only consider photons in the equatorial plane which are in outward radial motion with respect to the ZAMOs, namely with 4-momentum

$$k = E(n)[n + \hat{\nu}(k, n)], \quad \hat{\nu}(k, n) = e_{\hat{r}},$$

where $E(n) = E/N$ is the relative energy of the photon and $E = -k_t$ is the conserved energy associated with the timelike Killing vector field and $L = k_\phi = 0$ is the vanishing conserved angular momentum associated with the rotational Killing vector field, while $\hat{\nu}(k, n)$ defines the unit vector direction of the relative velocity. For the Schwarzschild case, these orbits are radial geodesics with respect to the static observers tied to the coordinate system, but for the Kerr case, they are dragged azimuthally by the rotation of the spacetime with respect to the coordinates.

$$T^{\alpha\beta}{}_{;\beta} = 0 \quad \rightarrow \quad \Phi = \frac{\Phi_0}{[g_{\theta\theta}g_{\phi\phi}]^{1/4}}$$

Test particle

Consider now a test particle moving in the equatorial plane $\theta = \pi/2$ accelerated by the radiation field, i.e., with 4-velocity

$$U = \gamma(U, n)[n + \nu(U, n)], \quad \nu(U, n) \equiv \nu^{\hat{r}} e_{\hat{r}} + \nu^{\hat{\phi}} e_{\hat{\phi}} = \nu \sin \alpha e_{\hat{r}} + \nu \cos \alpha e_{\hat{\phi}}$$

where $\gamma(U, n) = 1/\sqrt{1 - \|\nu(U, n)\|^2}$ is the Lorentz factor and the abbreviated notation $\nu^{\hat{a}} = \nu(U, n)^{\hat{a}}$ has been used. In a similarly abbreviated notation, $\nu = \|\nu(U, n)\|$ and α are the magnitude of the spatial velocity $\nu(U, n)$ and its polar angle measured clockwise from the positive ϕ direction in the r - ϕ tangent plane, while $\hat{\nu} = \hat{\nu}(U, n)$ is the associated unit vector. Note that $\alpha = 0$ corresponds to azimuthal motion with respect to the ZAMOs, while $\alpha = \pm\pi/2$ corresponds to (outward/inward) radial motion with respect to the ZAMOs.

$$\begin{aligned} a(U)^{\hat{t}} &= \gamma^2 \nu \sin \alpha \left[a(n)^{\hat{r}} + 2\nu \cos \alpha \theta(n)^{\hat{r}}_{\hat{\phi}} \right] + \gamma^3 \nu \frac{d\nu}{d\tau} \\ a(U)^{\hat{r}} &= \gamma^2 [a(n)^{\hat{r}} + k_{(\text{lie})}(n)^{\hat{r}} \nu^2 \cos^2 \alpha + 2\nu \cos \alpha \theta(n)^{\hat{r}}_{\hat{\phi}}] \\ &\quad + \gamma \left[\gamma^2 \sin \alpha \frac{d\nu}{d\tau} + \nu \cos \alpha \frac{d\alpha}{d\tau} \right] \end{aligned}$$

$$a(U)^{\hat{\theta}} = 0,$$

$$a(U)^{\hat{\phi}} = -\gamma^2 \nu^2 \sin \alpha \cos \alpha k_{(\text{lie})}(n)^{\hat{r}} + \gamma \left(\gamma^2 \cos \alpha \frac{d\nu}{d\tau} - \nu \sin \alpha \frac{d\alpha}{d\tau} \right)$$

4-acceleration frame components

Motion equations

Scattering of radiation as well as the momentum-transfer cross section σ (assumed to be a constant) of the particle is independent of the direction and frequency of the radiation; therefore the associated force is given by

$$\mathcal{F}_{(\text{rad})}(U)^\alpha = -\sigma P(U)^\alpha{}_\beta T^\beta{}_\mu U^\mu$$

where $P(U)^\alpha{}_\beta = \delta^\alpha_\beta + U^\alpha U_\beta$ projects orthogonally to U . The equation of motion of the particle then becomes

$$ma(U) = \mathcal{F}_{(\text{rad})}(U),$$

where m is the mass of the particle and $a(U) = \nabla_U U$ is its 4-acceleration.

$$\frac{d\nu}{d\tau} = -\frac{\sin \alpha}{\gamma} [a(n)^{\hat{r}} + 2\nu \cos \alpha \theta(n)^{\hat{r}}{}_{\hat{\phi}}] + \frac{A}{N^2 \sqrt{g_{\theta\theta} g_{\phi\phi}}} (1 - \nu \sin \alpha) (\sin \alpha - \nu),$$

$$\begin{aligned} \frac{d\alpha}{d\tau} = & -\frac{\gamma \cos \alpha}{\nu} [a(n)^{\hat{r}} + 2\nu \cos \alpha \theta(n)^{\hat{r}}{}_{\hat{\phi}} + \nu^2 k_{(\text{lie})}(n)^{\hat{r}}] \\ & + \frac{A}{N^2 \sqrt{g_{\theta\theta} g_{\phi\phi}}} \frac{(1 - \nu \sin \alpha) \cos \alpha}{\nu}, \end{aligned}$$

$$\frac{dr}{d\tau} = \frac{\gamma \nu \sin \alpha}{\sqrt{g_{rr}}},$$

where the positive constant A is defined by

$$A = \sigma \Phi_0^2 E^2$$

Equations of motion,
to be integrated numerically.

Explicit spacetime examples

The Schwarzschild spacetime is characterized by the metric functions

$$g_{tt} = -N^2, \quad g_{t\phi} = 0, \quad g_{rr} = 1/N^2, \quad g_{\theta\theta} = r^2, \quad g_{\phi\phi} = r^2 \sin^2 \theta,$$

where the lapse function is

$$N = \sqrt{1 - \frac{2M}{r}} \sim 1 - \frac{M}{r},$$

in which the approximate expression represents the asymptotic value at $r \rightarrow \infty$ and to first order in M . In this case as well the ZAMOs are aligned with the coordinate time world lines; however, they form an accelerated ($a(n)^{\hat{r}} = M/(r\sqrt{r^2 - 2Mr}) \sim M/r^2$) and expansionfree ($\theta(n) = 0$) congruence. The Lie curvature of the ϕ loops has only a radial component with value $k_{(\text{lie})}(n)^{\hat{r}} = -N/r \sim -1/r + M/r^2$. Radially outgoing (geodesic) photons on the equatorial plane have 4-momentum

$$k = E \left[\left(1 - \frac{2M}{r} \right)^{-1} \partial_t + \partial_r \right] = E(n)[n + e_{\hat{r}}].$$

Eqs. (2.29) reduce to

$$\begin{aligned} \frac{d\nu}{d\tau} &= \frac{\sin \alpha}{\gamma} k_{(\text{lie})}(n)^{\hat{r}} \nu_K^2 + \frac{A}{r^2 N^2} (1 - \nu \sin \alpha) (\sin \alpha - \nu), \\ \frac{d\alpha}{d\tau} &= -\frac{\gamma \cos \alpha}{\nu} k_{(\text{lie})}(n)^{\hat{r}} (\nu^2 - \nu_K^2) + \frac{A}{r^2 N^2 \nu} (1 - \nu \sin \alpha) \cos \alpha, \\ \frac{dr}{d\tau} &= \gamma \nu N \sin \alpha, \end{aligned}$$

where we have used the relation $a(n)^{\hat{r}} = -k_{(\text{lie})}(n)^{\hat{r}} \nu_K^2$,

$$\nu_K = \sqrt{\frac{M}{r - 2M}}$$

is the Keplerian speed associated with circular geodesics. If one is only interested in the spatial orbit of the particle, one can choose r or ϕ as the parameter along its path, re-expressing the above equations using the chain rule.

For the case $\nu = 0$ of a particle at rest, these equations reduce to the single condition Eq. (2.33) representing the balancing of the gravitational attraction and the radiation pressure at constant r and ϕ , namely

$$\frac{A}{M} = \left(1 - \frac{2M}{r} \right)^{1/2} \rightarrow r = r_{(\text{crit})} \equiv \frac{2M}{1 - A^2/M^2}.$$

Schwarzschild

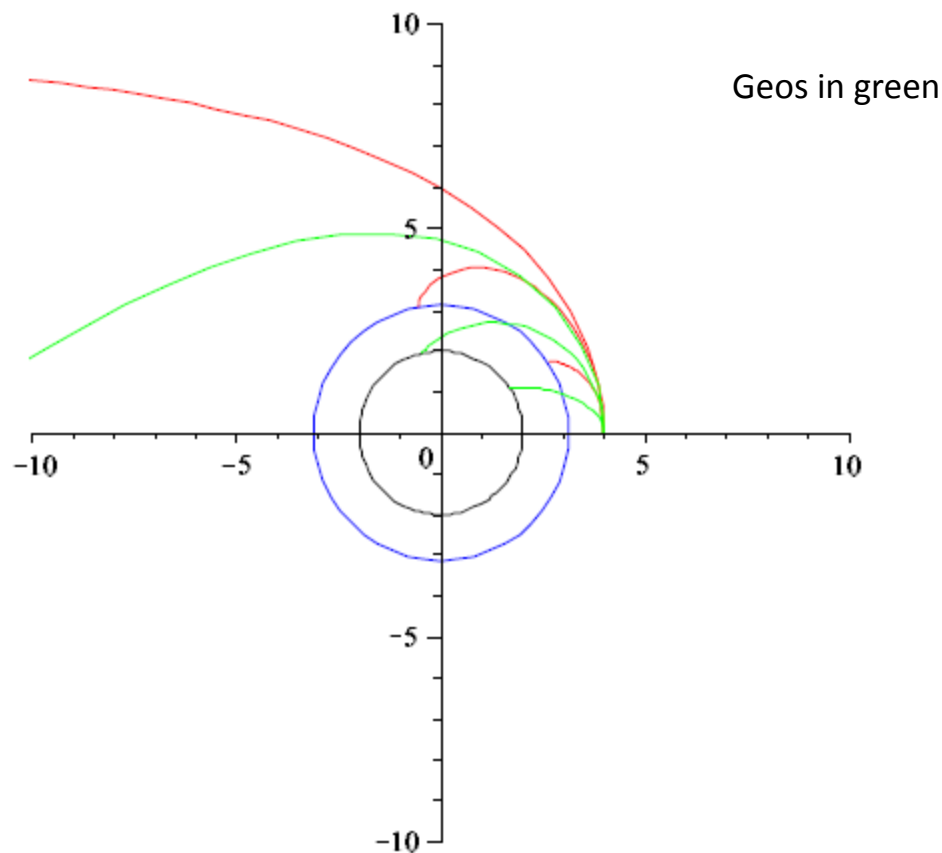


Figure 1. The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/M = 0.6$, $r_{\text{crit}} = 3.125M$. The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius which is inside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$. The corresponding geodesics $A/M = 0$ are in gray.

Geos in green

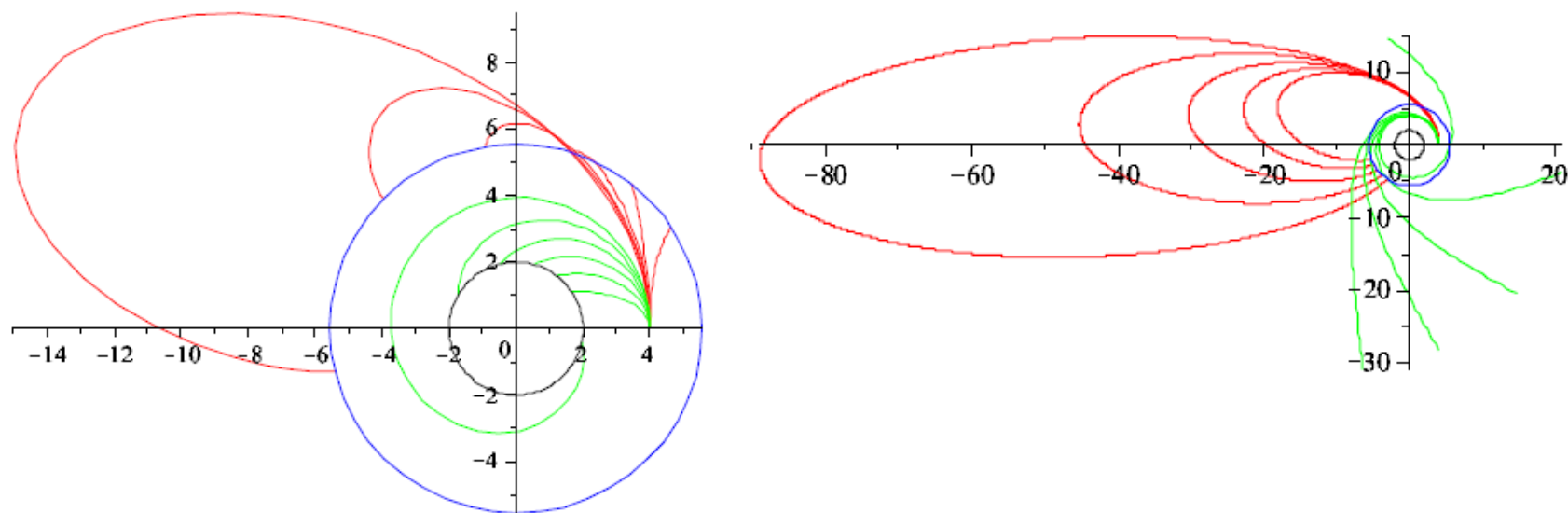


Figure 2. The orbit of the particle in the Schwarzschild spacetime with $M = 1$, $A/M = 0.8$, $r_{\text{crit}} = 5.5M$. The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius which is outside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and for the left figure $\nu(0) = 0.2, 0.3, \dots, 0.7$ while for the right figure $0.71, 0.72, \dots, 0.75$. The corresponding geodesics $A/M = 0$ are in gray.

In the equatorial plane of the Kerr metric, the metric is

$$g_{tt} = -\left(1 - \frac{2M}{r}\right), \quad g_{t\phi} = -\frac{2aM}{r}, \quad g_{\phi\phi} = \frac{r^3 + a^2r + 2a^2M}{r},$$

$$g_{rr} = \frac{r^2}{\Delta}, \quad g_{\theta\theta} = r^2,$$

so that

$$N = \sqrt{\frac{r\Delta}{r^3 + a^2r + 2a^2M}} \sim 1 - \frac{M}{r}, \quad N^\phi = -\frac{2aM}{r^3 + a^2r + 2a^2M} \sim -\frac{2aM}{r^3}$$

where $\Delta = r^2 + a^2 - 2Mr$ and the approximate expressions represent their asymptotic values ($r \rightarrow \infty$) to first order in M . The ZAMOs are timelike outside the horizon

$r_+ = M + \sqrt{M^2 - a^2}$. The nonvanishing components of the ZAMO kinematical fields are

$$a(n)^{\hat{r}} = \frac{M[(r^2 + a^2)^2 - 4a^2Mr]}{r^2\sqrt{\Delta}(r^3 + a^2r + 2a^2M)} \sim \frac{M}{r^2},$$

$$\theta(n)^{\hat{r}}_{\hat{\phi}} = -\frac{aM(3r^2 + a^2)}{r^2(r^3 + a^2r + 2a^2M)} \sim -\frac{3aM}{r^3},$$

$$k_{(\text{lie})}(n)^{\hat{r}} = -\frac{\sqrt{\Delta}(r^3 - a^2M)}{r^2(r^3 + a^2r + 2a^2M)} \sim -\frac{1}{r} + \frac{M}{r^2}.$$

Circular geodesics correspond to orbits

$$U_\pm = \gamma_\pm(n + \nu_\pm e_{\hat{\phi}}), \quad \nu_\pm = \frac{a^2 \mp 2a\sqrt{Mr} + r^2}{\sqrt{\Delta}(a \pm r\sqrt{r/M})},$$

and the following relation between $\nu_\pm$ and the ZAMO kinematical fields hold

$$a(n)^{\hat{r}} = k_{(\text{lie})}(n)^{\hat{r}}\nu_+\nu_-, \quad -2\theta(n)^{\hat{r}}_{\hat{\phi}} = k_{(\text{lie})}(n)^{\hat{r}}(\nu_+ + \nu_-).$$

Equations (2.29) can then be rewritten as

$$\frac{d\nu}{d\tau} = -\frac{\sin\alpha k_{(\text{lie})}(n)^{\hat{r}}}{\gamma}[\nu_+\nu_- - \nu\cos\alpha(\nu_+ + \nu_-)] + \frac{A(1 - \nu\sin\alpha)(\sin\alpha - \nu)}{r\sqrt{g_{\phi\phi}}N^2},$$

$$\frac{d\alpha}{d\tau} = -\frac{\gamma\cos\alpha k_{(\text{lie})}(n)^{\hat{r}}}{\nu}[\nu_+\nu_- - \nu\cos\alpha(\nu_+ + \nu_-) + \nu^2] + \frac{A(1 - \nu\sin\alpha)\cos\alpha}{r\sqrt{g_{\phi\phi}}N^2\nu},$$

$$\frac{dr}{d\tau} = \frac{\gamma\nu\sin\alpha}{\sqrt{g_{rr}}}.$$

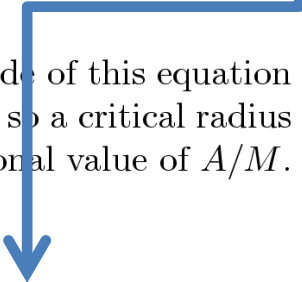
Kerr

For the case $\nu = 0$ of a particle remaining at rest with respect to the ZAMOs, these reduce to the single radial force balance condition

$$\frac{A}{M} = \frac{[(r^2 + a^2)^2 - 4a^2Mr]\sqrt{\Delta}}{r^2(g_{\phi\phi})^{3/2}},$$

which cannot be solved explicitly for r . However, the right hand side of this equation takes values between 0 at the horizon $r = r_+$ and 1 when as $r \rightarrow \infty$ so a critical radius $r_{\text{(crit)}}$ always exists for which this is satisfied for any proper fractional value of A/M . If $A/M > 1$ of course no static solutions exist.

Kerr



i.e. it rotates wrt to the coords with the ZAMO angular velocity

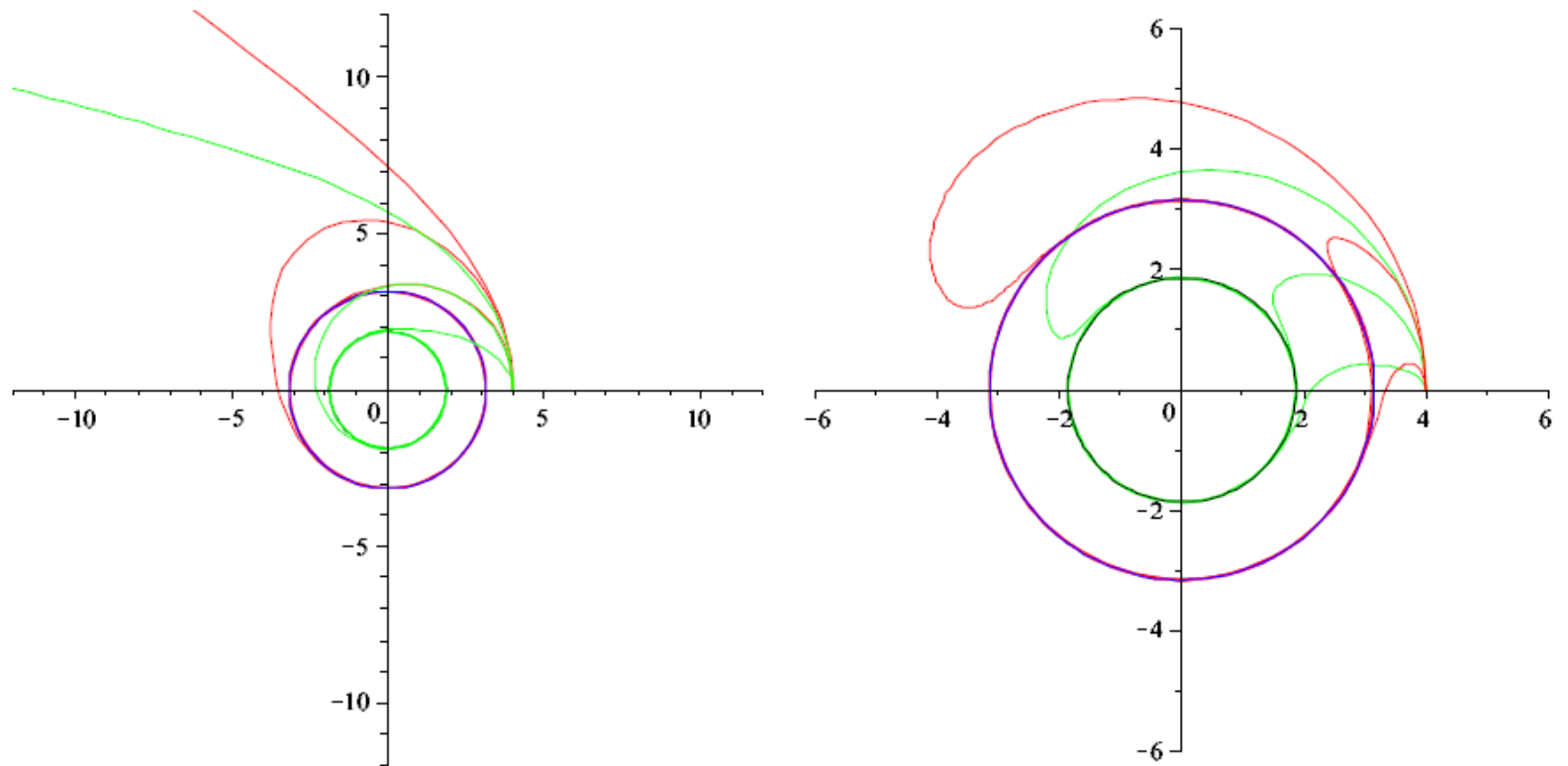


Figure 3. The orbit of the particle in the Kerr spacetime with $M = 1$, $a = 0.5$ (left figure), $a = -0.5$ (right figure), $A/M = 0.6$, $r_{\text{crit}} = 3.154M$. The inner circle is the horizon $r = 1.866M$, while the outer circle is at the critical radius which is inside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$. The corresponding geodesics $A/M = 0$ are in gray.

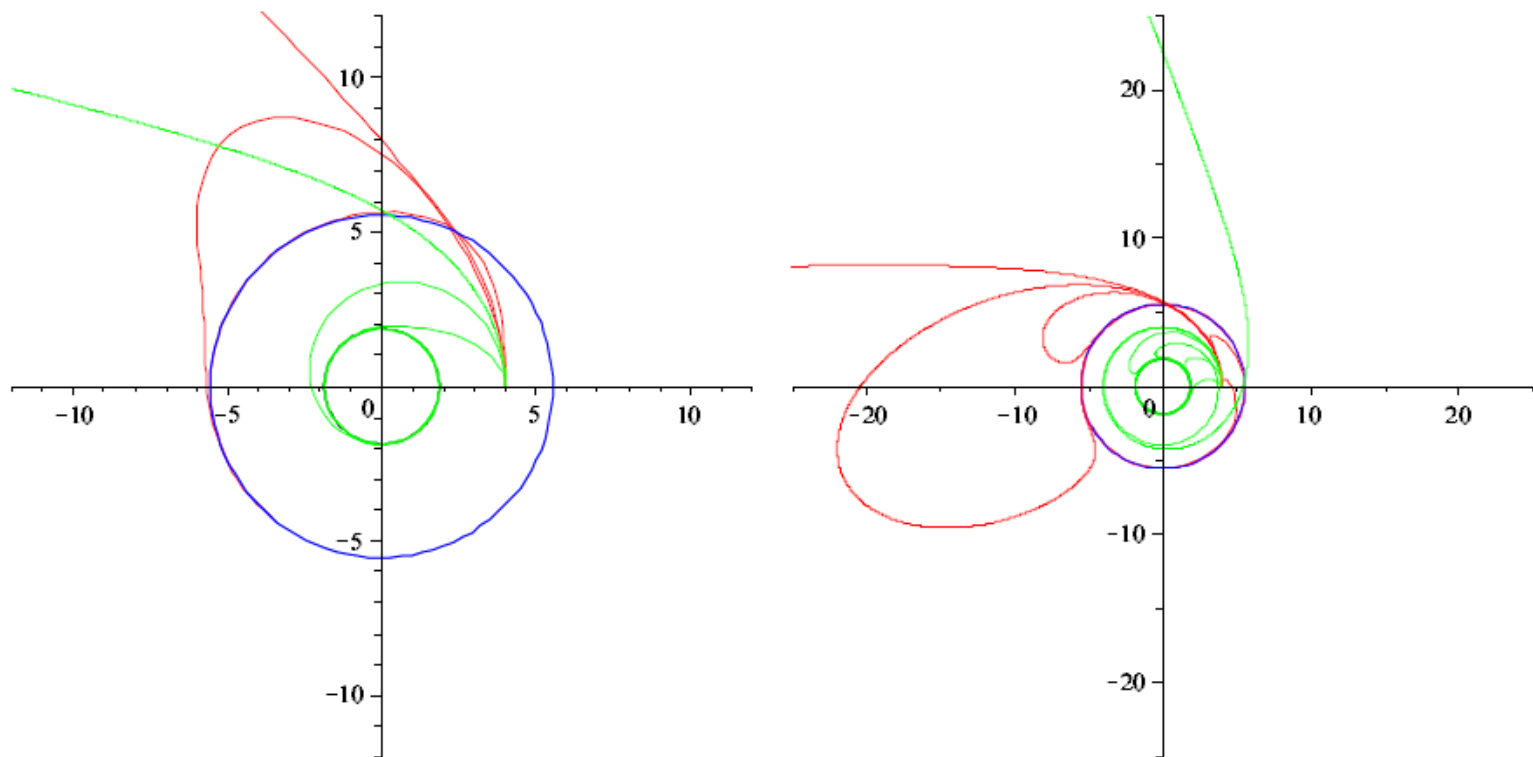


Figure 4. The orbit of the particle in the Kerr spacetime with $M = 1$, $a = 0.5$ (left figure), $a = -0.5$ (right figure), $A/M = 0.8$, $r_{\text{crit}} = 5.551M$. The inner circle is the horizon $r = 1.866M$, while the outer circle is at the critical radius which is outside the initial data position. Initial conditions have $(r(0), \phi(0), \alpha(0)) = (4M, 0, 0)$ and $\nu(0) = 0.2, 0.5, 0.8$ for the left figure, while in the right figure $\nu(0) = 0.2, 0.5, 0.8, 0.847$ for both the accelerated and geodesic curves and then finally $\nu(0) = 0.9$ for the accelerated curve and $\nu(0) = 0.848$ for the geodesic, both of which escape to infinity. The corresponding geodesics $A/M = 0$ are in gray.

Generalizations



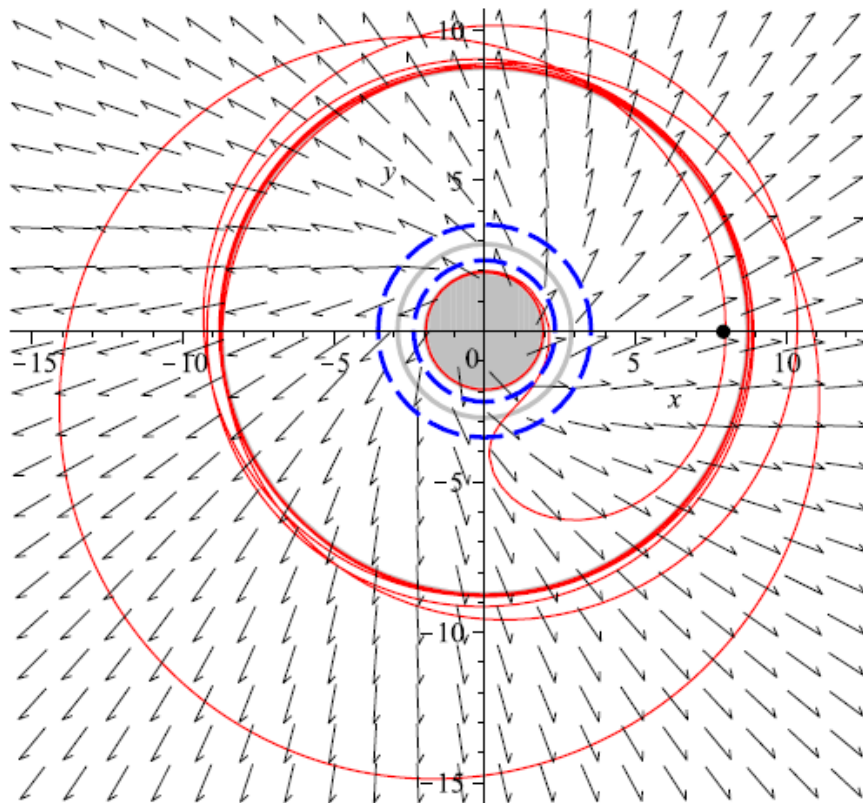
Photons with nonzero angular momentum

Radiation field as an exact solution of the Einstein's field equations
(Vaidya spacetime)

Test body endowed with intrinsic angular momentum

Finite size of the emitting star

Etc.



$$b=L/E$$

The spacetime and photon parameters are $a = 0.5$, $A/M = 0.3$, $b/M = 3$, showing two orbits moving initially from the bullet point in the two azimuthal directions just inside the outer critical radius with 1.2 times the critical speed for that counterclockwise critical orbit. The unit velocity direction field $\hat{v}(k, n)$ of the radiation with respect to the ZAMOs is superimposed on the plot. The dashed circles are the two null circular geodesics orbits. The gray filled circle extends to the horizon. The clockwise moving orbit settles down to the critical orbit, while the counterclockwise moving orbit quickly falls into the inner critical orbit near the horizon. The gray circle between the null orbits is the unstable critical orbit. This and all further plots show r/M on the axes.

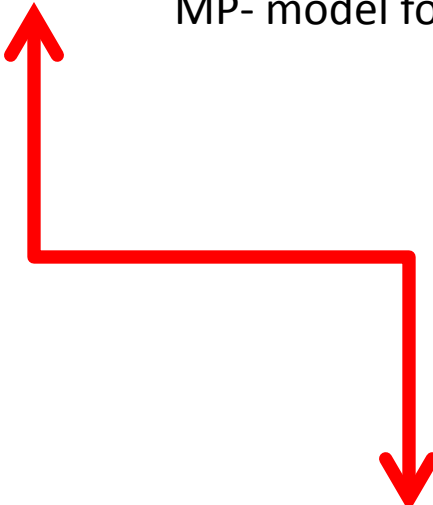
Generalization to the case of spinning test particles undergoing PR effect

$$\frac{DP^\mu}{d\tau} = -\frac{1}{2}R^\mu{}_{\nu\alpha\beta}U^\nu S^{\alpha\beta} \equiv F^{(\text{spin})\mu}$$

$$\frac{DS^{\mu\nu}}{d\tau} = P^\mu U^\nu - P^\nu U^\mu,$$

$$S^{\mu\nu}P_\nu = 0$$

MP- model for spinning test particles

$$\frac{DP^\mu}{d\tau} = F^{(\text{spin})\mu} + F^{(\text{rad})\mu}$$


Let the 4-momentum $P = mu$ be fully general, i.e. with

$$u = \gamma_u [n + \nu_u (\sin \alpha_u e_{\hat{r}} + \cos \alpha_u e_{\hat{\phi}})] , \quad \gamma_u = \frac{1}{\sqrt{1 - \nu_u^2}} ,$$

and introduce the spin vector associated with $S_{\mu\nu}$ by spatial duality

$$S^\beta = u_\alpha \eta^{\alpha\beta\mu\nu} S_{\mu\nu} ,$$

where $\eta_{\alpha\beta\gamma\delta} = \sqrt{-g} \epsilon_{\alpha\beta\gamma\delta}$ is the unit volume 4-form and $\epsilon_{\alpha\beta\gamma\delta}$ ($\epsilon_{0123} = 1$) is the Levi-Civita alternating symbol. It is also useful to consider the scalar invariant

$$s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu} ,$$

constant along $\mathcal{C}_U$ because of Eqs. (1.2) and (1.3). Consistency of the model requires that the length scale $|s|/m$ associated with the spinning particle be much smaller than the one associated with the background spacetime, say M , namely

$$|\hat{s}| \equiv \frac{|s|}{mM} \ll 1 .$$

Let us consider Eqs. (1.1) and (1.2) with U^α given by Eq. (2.7) and u^α given by Eq. (2.14). In the spinless case P is aligned with U , i.e. $u = U$, implying that $\nu = \nu_u$. The presence of the spin causes a change in both U and u according to

$$U = U_0 + \hat{s} U_{\hat{s}} , \quad u = U_0 + \hat{s} u_{\hat{s}} ,$$

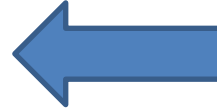
where

$$U_0 = \gamma_0 (n + \nu_0^{\hat{r}} e_{\hat{r}} + \nu_0^{\hat{\phi}} e_{\hat{\phi}})$$

satisfies Eq. (2.12) and corrections are first order in the spin. Higher order terms in Eqs. (1.1) and (1.2) are neglected. This leads to two different sets of equations for zeroth and first order in spin respectively, which are listed in Appendix A.

The spin force to first order in $\hat{s}$ is given by

$$F^{(\text{spin})} = -\frac{3mM^2}{r_0^3} \hat{s} \gamma_0^2 \nu_0^{\hat{\phi}} (\nu_0^{\hat{r}} n + e_{\hat{r}}) .$$



We find that the mass of the spinning particle m is a constant of motion. Furthermore, from the evolution equations for the spin it follows that the spin vector has a single nonvanishing and constant component along θ (or z), namely

$$S = -S^{\hat{\theta}} e_{\hat{\theta}} = s e_z .$$

It is convenient to introduce a friction parameter f , so that the length scale A associated with the radiation field be much smaller than M , i.e.

$$f \equiv \frac{A}{M} \ll 1.$$

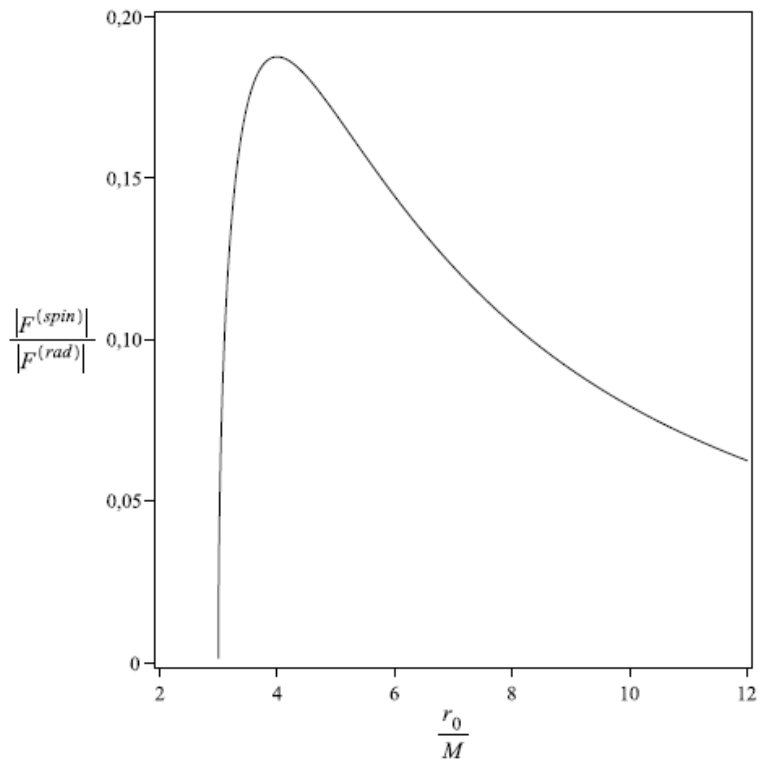


Figure 4. The ratio between the magnitudes of spin and radiation forces given by Eq. (3.13) is plotted in units of $|\hat{s}|/f$ as a function of r_0/M .

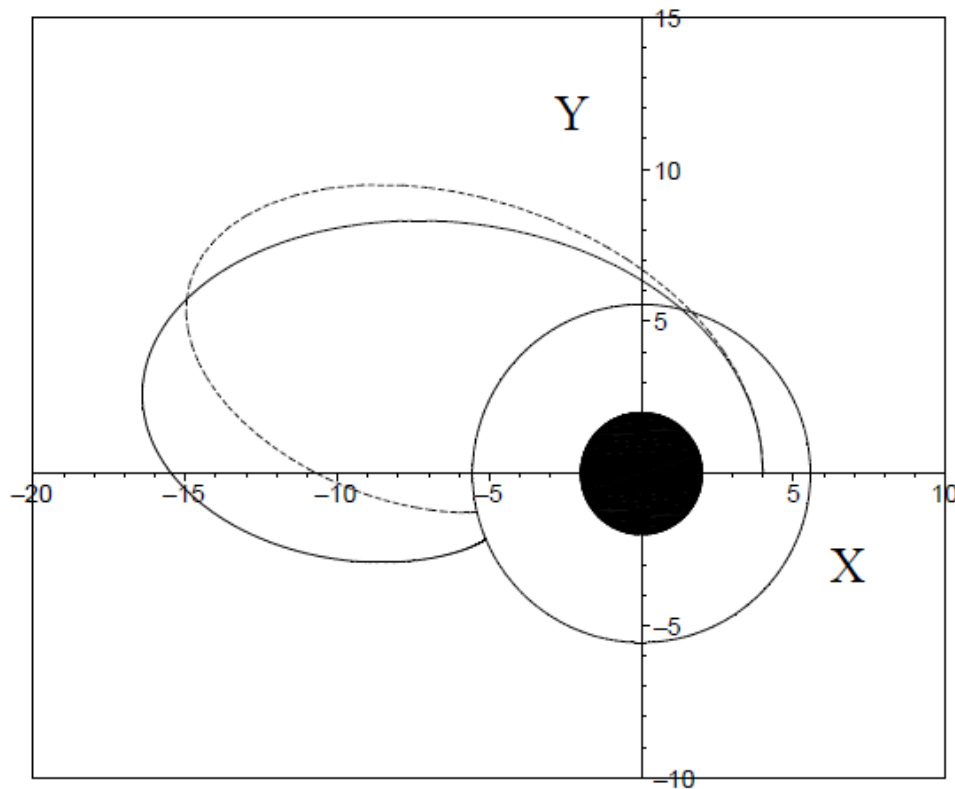


Figure 1. The orbit of a spinning particle (solid curve) undergoing Poynting-Robertson effect is shown for the choice of parameters $A/M = 0.8$ and $\hat{s} = 0.5$ ($X = r \cos \phi$ and $Y = r \sin \phi$ are Cartesian-like coordinates). The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $\nu_{u0}(0) = 0.7$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $\nu_s^{\hat{r}}(0) = 0$ and $\nu_s^{\hat{\phi}}(0) = 0$. The values of the spin parameter has been exaggerated in order to evidentiate the difference from the motion of a spinless particle (dashed curve). The inner circle is the horizon $r = 2M$, while the outer circle is at the critical radius $r_{\text{crit}} = 5.5M$ which is outside the initial data position.

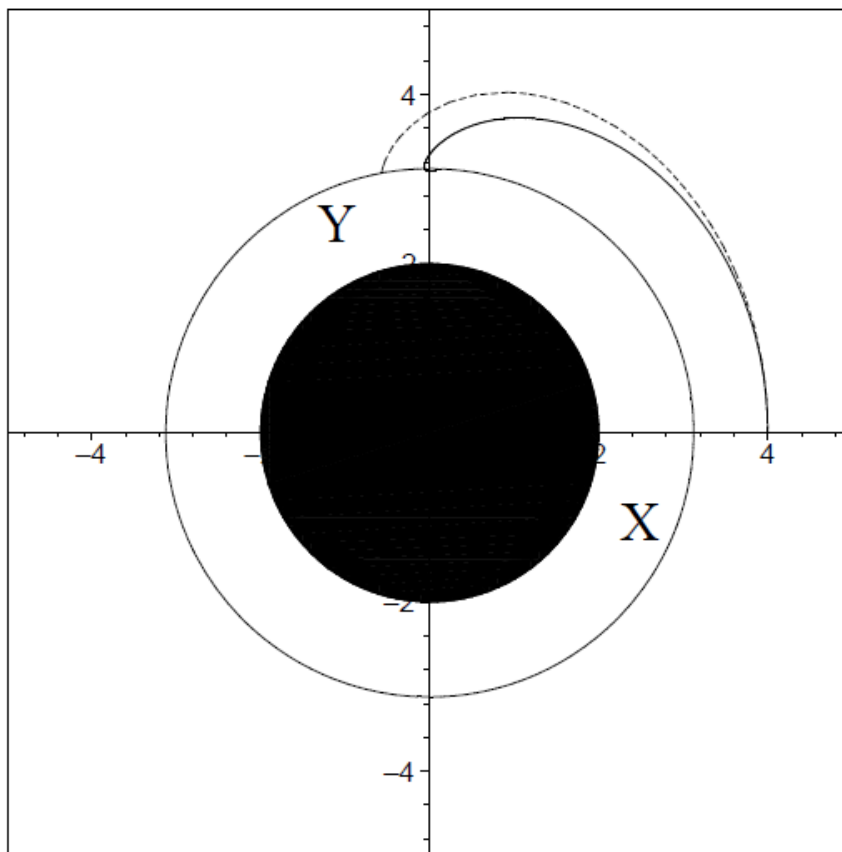


Figure 2. The orbit of a spinning particle (solid curve) undergoing Poynting-Robertson effect is shown for the choice of parameters $A/M = 0.6$ and $\hat{s} = 0.5$. The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $\nu_{u0}(0) = 0.5$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $\nu_s^{\hat{r}}(0) = 0$ and $\nu_s^{\hat{\phi}}(0) = 0$. The corresponding orbit for a spinless particle is also shown (dashed curve). The critical radius $r_{\text{crit}} = 3.125M$ is inside the initial data position.

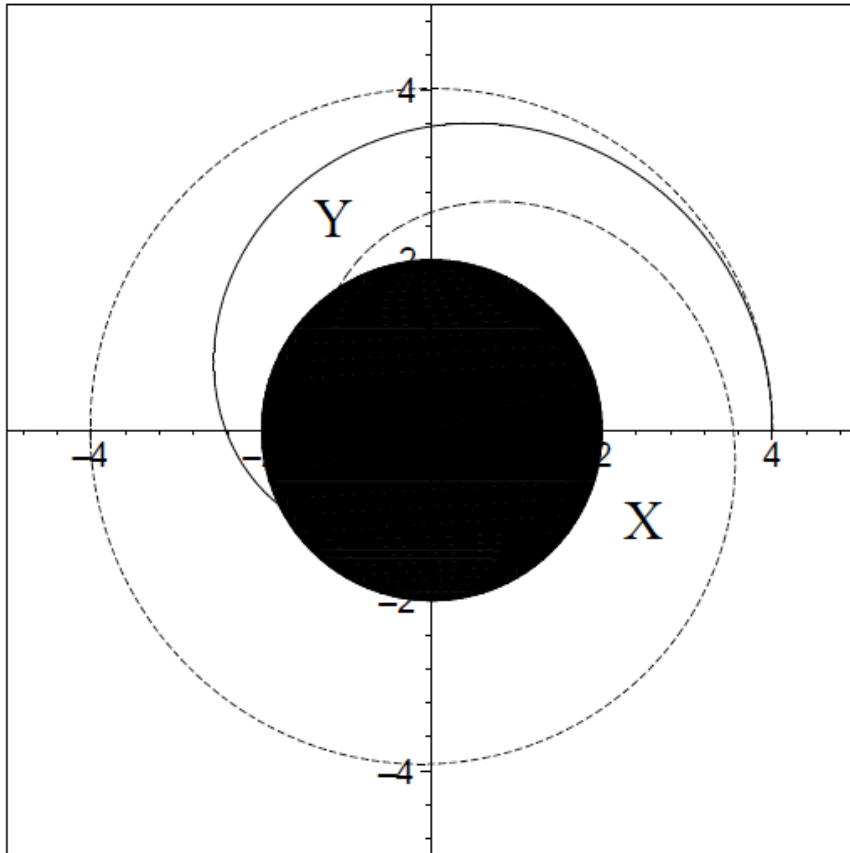


Figure 3. The orbit of a spinning particle (solid curve) undergoing Poynting-Robertson effect is shown for the choice of parameters $A/M = 0.01$ and $\hat{s} = 0.5$. The starting point is located at $r_0(0) = 4M$ and $\phi_0(0) = 0$ with $\nu_{u0}(0) = \nu_K \approx 0.7071$, $\alpha_{u0}(0) = 0$, $t_s(0) = 0$, $r_s(0) = 0$ and $\phi_s(0) = 0$, $\nu_{\hat{r}s}(0) = 0$ and $\nu_{\hat{\phi}s}(0) = 0$. The corresponding orbit for a spinless particle is also shown (dashed curve). In this case $r_{(\text{crit})} \approx 2M$.

Details...



Deviation from the circular geodesic

Consider now the corrections to geodesic circular motion, by taking the effect of the radiation field also small.

In absence of both spin and radiation we assume the geodesic motion of the particle to be circular at $r = r_0$ ($r_0 > 3M$ in order U_K to be timelike), that is

$$U = U_K = \gamma_K (n \pm \nu_K e_{\hat{\phi}}), \quad (3.1)$$

where the Keplerian value of speed (ν_K) and the associated Lorentz factor (γ_K) and angular velocity (ζ_K) are given by

$$\nu_K = \sqrt{\frac{M}{r_0 - 2M}}, \quad \gamma_K = \sqrt{\frac{r_0 - 2M}{r_0 - 3M}}, \quad \zeta_K = \sqrt{\frac{M}{r_0^3}}. \quad (3.2)$$

The $\pm$ signs in Eq. (3.1) correspond to co-rotating (+) or counter-rotating (−) orbits with respect to increasing values of the azimuthal coordinate ϕ (counter-clockwise

motion as seen from above). The azimuthal direction in the local rest space of U_K pointing in the direction of relative motion (i.e. the boost of $e_{\hat{\phi}}$ in the local rest space of U_K) is specified by the following unit vector orthogonal to U_K in the t - ϕ plane

$$\bar{U}_K = \gamma_K (\nu_K n \pm e_{\hat{\phi}}),$$

where the $\pm$ signs are correlated with those in U_K .

The parametric equations of U_K are

$$\begin{aligned} t_K &= t_0 + \frac{\gamma_K}{N_0} \tau \equiv t_0 + \Gamma_K \tau, \\ r &= r_0, \quad \theta = \frac{\pi}{2}, \\ \phi_K &= \phi_0 \pm \frac{\gamma_K \nu_K}{r_0} \tau \equiv \phi_0 \pm \Omega_K \tau, \end{aligned}$$

where now t_0 , r_0 and ϕ_0 are constants and

$$\Gamma_K = \sqrt{\frac{r_0}{r_0 - 3M}}, \quad \Omega_K = \frac{1}{r_0} \sqrt{\frac{M}{r_0 - 3M}}.$$

It is convenient to introduce a friction parameter f , so that the length scale A associated with the radiation field be much smaller than M , i.e.

$$f \equiv \frac{A}{M} \ll 1.$$

Therefore, in the present analysis corrections to geodesic motion will be limited to first order terms in both parameters $\hat{s}$ and f , according to

$$\begin{aligned} t &= t_K + f t_f + \hat{s} t_{\hat{s}}, \quad r = r_0 + f r_f + \hat{s} r_{\hat{s}}, \quad \phi = \phi_K + f \phi_f + \hat{s} \phi_{\hat{s}}, \\ \nu^{\hat{t}} &= f \nu_f^{\hat{t}} + \hat{s} \nu_{\hat{s}}^{\hat{t}}, \quad \nu^{\hat{\phi}} = \pm \nu_K + f \nu_f^{\hat{\phi}} + \hat{s} \nu_{\hat{s}}^{\hat{\phi}}, \\ \nu_u &= \pm \nu_K + f \nu_{uf} + \hat{s} \nu_{u\hat{s}}, \quad \alpha_u = f \alpha_{uf} + \hat{s} \alpha_{u\hat{s}}, \end{aligned}$$

where t_K and ϕ_K are given by Eq. (3.4). This implies

$$U = U_K + f U_f + \hat{s} U_{\hat{s}},$$

where

$$\begin{aligned} U_f &= \left(-\nu_K \frac{r_f}{r_0} \pm \gamma_K^2 \nu_f^{\hat{\phi}} \right) U_K + \gamma_K \nu_f^{\hat{t}} e_{\hat{t}}, \\ U_{\hat{s}} &= \left(-\nu_K \frac{r_{\hat{s}}}{r_0} \pm \gamma_K^2 \nu_{\hat{s}}^{\hat{\phi}} \right) \bar{U}_K + \gamma_K \nu_{\hat{s}}^{\hat{t}} e_{\hat{t}}. \end{aligned}$$

Similarly

$$u = U + f u_f + \hat{s} u_{\hat{s}},$$

with U given by Eq. (3.8) and

$$u_f = \gamma_K (\nu_K^2 \nu_f^{\hat{\phi}} - \nu_f^{\hat{t}}) e_{\hat{t}}, \quad u_{\hat{s}} = \gamma_K (\nu_K^2 \nu_{\hat{s}}^{\hat{\phi}} - \nu_{\hat{s}}^{\hat{t}}) e_{\hat{t}},$$

as discussed in Appendix B.

To first order in $\hat{s}$ and f the spin force and radiation force are given by

$$\begin{aligned} F^{(\text{spin})} &= \mp 3mM \hat{s} \gamma_K^2 \zeta_K^2 \nu_K e_{\hat{t}}, \\ F^{(\text{rad})} &= -mf \Omega_K \nu_K (\gamma_K \nu_K U_K - e_{\hat{t}}), \end{aligned}$$

$$r = r_0 + f r_f + \hat{s} r_{\hat{s}}$$



$$t_{\delta} = \mp \frac{6M^2}{r_0} \frac{\Omega_K^3}{\Omega_{\text{ep}}^3} [\sin(\Omega_{\text{ep}}\tau) - \Omega_{\text{ep}}\tau],$$

$$t_f = 4r_0\zeta_K\nu_K^2 \frac{\Omega_K^3}{\Omega_{\text{ep}}^4} \left\{ [\cos(\Omega_{\text{ep}}\tau) - 1] + \frac{\Omega_{\text{ep}}}{2r_0\zeta_K\Omega_K} [\sin(\Omega_{\text{ep}}\tau) - \Omega_{\text{ep}}\tau] + \frac{3}{8}\gamma_K^2\Omega_{\text{ep}}^2\tau^2 \right\}$$

$$r_{\delta} = \pm 3r_0 \frac{\Omega_K\zeta_K}{\Omega_{\text{ep}}^2} [\cos(\Omega_{\text{ep}}\tau) - 1],$$

$$r_f = -r_0\zeta_K \frac{\Omega_K}{\Omega_{\text{ep}}^2} \left\{ [\cos(\Omega_{\text{ep}}\tau) - 1] - 2r_0\zeta_K \frac{\Omega_K}{\Omega_{\text{ep}}} [\sin(\Omega_{\text{ep}}\tau) - \Omega_{\text{ep}}\tau] \right\},$$

$$\phi_{\delta} = \pm \frac{\zeta_K}{\nu_K^2} t_{\delta}, \quad \phi_f = \pm \frac{\zeta_K}{\nu_K^2} t_f,$$

where

$$\Omega_{\text{ep}} = \sqrt{\frac{M(r_0 - 6M)}{r_0^3(r_0 - 3M)}}$$

is the well known epicyclic frequency governing the radial perturbations of circular geodesics.

The constant term in r_{δ} represents the slight change in the radius of the circular orbit about which the solution oscillates with proper period $2\pi/\Omega_{\text{ep}}$. In contrast, the presence of a secular term in r_f is responsible for the deviation from geodesic motion due to friction, which is measurable in principle. In fact, by taking the mean values over a period of the perturbed radius we can estimate the amount of variation of radial distance

$$\left\langle \frac{\delta r}{r} \right\rangle \equiv \frac{r - r_0}{r_0} = \Gamma_K \frac{\zeta_K^2}{\Omega_{\text{ep}}^2} \left[\left(1 - 2\pi r_0 \zeta_K \frac{\Omega_K}{\Omega_{\text{ep}}} \right) f \mp 3M \hat{s} \gamma_K \zeta_K N_0 \right].$$



Numbers

For instance, for the motion of the Earth about the Sun we find


$$\left\langle \frac{\delta r}{r} \right\rangle \approx f \mp 2 \times 10^{-17} \frac{(s/m)_{\oplus}}{\text{cm}} \approx 3 \times 10^{-5} \mp 4 \times 10^{-15} \approx 3 \times 10^{-5}, \quad (3.18)$$

since $r_0 \approx 1.5 \times 10^{13}$ cm, $M = M_{\odot} \approx 1.5 \times 10^5$ cm and the ratio $(s/m)_{\oplus} \approx 200$ cm for the Earth; the friction parameter is related to the ratio between the solar luminosity $\mathcal{L}_{\odot} \approx 3.8 \times 10^{33}$ erg/s and the Eddington luminosity [20, 21] $\mathcal{L}_{\text{Edd}} \approx 1.3 \times 10^{38}$ erg/s, and for the Sun is given by $f \approx 3 \times 10^{-5}$. Therefore, in this case the effect of the radiation field on the orbit dominates. Note that the estimate of the contribution due to spin is in agreement with [22].



The effect of the spin may become important when considering the orbiting extended body is a fast rotating object. To illustrate the order of magnitude of the effect, we may consider the binary pulsar system PSR J0737-3039 as orbiting Sgr A*, the supermassive ($M \simeq 10^6 M_{\odot}$) black hole located at the Galactic Center [23, 24], at a distance of $r \simeq 10^9$ Km. The PSR J0737-3039 system consists of two close neutron stars (their separation is only $d_{AB} \sim 8 \times 10^5$ Km) of comparable masses $m_A \simeq 1.4 M_{\odot}$, $m_B \simeq 1.2 M_{\odot}$), but very different intrinsic spin period (23 ms of pulsar A vs 2.8 s of pulsar B) [25]. Its orbital period is about 2.4 hours, the smallest yet known for such an object. Since the intrinsic rotations are negligible with respect to the orbital period, we can treat the binary system as a single object with reduced mass $\mu_{AB} \simeq 0.7 M_{\odot}$ and intrinsic rotation equal to the orbital period. The spin parameter thus turns out to be equal to $\hat{s} \approx 1.0 \times 10^{-3}$. The luminosity of Sgr A* is about $10^3 \mathcal{L}_{\odot}$, whereas its Eddington luminosity is $\mathcal{L}_{\text{Edd}} \approx 10^{11} \mathcal{L}_{\odot}$, so that $f \approx 3 \times 10^{-8}$. Therefore, in this case

$$\left\langle \frac{\delta r}{r} \right\rangle \approx 7.6 \times 10^{-9} \mp 1.8 \times 10^{-7} \approx 1.7 \times 10^{-7}. \quad (3.19)$$

References

The general relativistic Poynting-Robertson effect

Bini D., Jantzen R. T., Stella L.

Classical and Quantum Gravity, vol. 26, 055009, 2009.

The general relativistic version is developed for Robertson's discussion of the Poynting-Robertson effect that he based on special relativity and Newtonian gravity for point radiation sources like stars. The general relativistic model uses a test radiation field of photons in outward radial motion with zero angular momentum in the equatorial plane of the exterior Schwarzschild or Kerrspacetime.

Spinning bodies undergoing Poynting-Robertson effect in the Schwarzschild spacetime

Bini D. and Geralico A.

To appear on Classical and Quantum Gravity, 2010

A spinning particle in the Schwarzschild spacetime deviates from geodesic behavior because of its spin.

A spinless particle also deviates from a geodesic behavior when a radiation field is superposed to the Schwarzschild background: in fact, the interaction with the radiation field, i.e. the absorbing and re-emitting radiation, is usually accounted by a friction-like force.

The latter is responsible for a well known effect already in Newtonian gravity, the Poynting-Robertson effect.

In this paper the Poynting-Robertson effect is extended to the case of spinning particles.

The Mathisson-Papapetrou model describing the motion of spinning test particles is suitably modified by accounting for the contribution of the radiation force. The resulting equations are numerically integrated and some typical orbits are shown in comparison with the spinless case.

Furthermore, the interplay between spin and radiation forces is discussed by analysing the deviation from circular geodesic motion on the equatorial plane when also the contribution due to friction can be treated as a small perturbation. Finally the estimate of the amount of radial variation from the geodesic radius has been shown to be measurable in principle.



The general relativistic Poynting-Robertson effect II. A general photon flux in black-hole space-times

Bini D., Geralico A., Jantzen R.T. , Semeràk O. and Stella L.

Submitted 2010

The Poynting-Robertson effect for a test particle orbiting a black hole in the presence of a stationary test radiation field with fixed angular momentum is analyzed in the context of a general stationary, axisymmetric and reflection symmetric spacetime, extending previous results for the case of a radiation field with zero angular momentum.

The specific cases of the Kerr and Schwarzschild black holes are discussed in detail, revealing a number of previously unexplored scenarios which are consistent with the equivalence principle.



Vaidya radiating metric and Poynting-Robertson effect

Bini D., Geralico A., Jantzen R.T. , Semeràk O.

Work in progress

GR tutorial for Dr. Maria Haney

Graduated from **Friedrich-Schiller-Universität Jena** in 2009.

GR formation: post-Newtonian dynamics and waveforms of bound and unbound compact binary systems, supervised by Prof. **Gerhard Schäfer**.

- ➡ How to use MapleTM as an advanced tool of symbolic calculus in GR
- ➡ Understanding GR from explicit spacetimes:
- ➡ Special orbits and frames in the Schwarzschild background
- ➡ Exact single GW spacetimes

A large number of exercises solved by Dr. Haney have led to a detailed analysis of an em radiation field superposed to the background of sandwich GW. Since most of these results are original they will be submitted for publication in the next months.

Thanks for your kind attention!

Donato Bini