QUANTUM STATES FROM TANGENT VECTORS*

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We argue that tangent vectors to classical phase space give rise to quantum states of the corresponding quantum mechanics. This is established for the case of complex, finite-dimensional, compact, classical phase spaces $\mathcal{C}$, by explicitly constructing Hilbert-space vector bundles over $\mathcal{C}$. We find that these vector bundles split as the direct sum of two holomorphic vector bundles: the holomorphic tangent bundle $T(\mathcal{C})$, plus a complex line bundle $N(\mathcal{C})$. Quantum states (except the vacuum) appear as tangent vectors to $\mathcal{C}$. The vacuum state appears as the fibrewise generator of $N(\mathcal{C})$. Holomorphic line bundles $N(\mathcal{C})$ are classified by the elements of $\mathrm{Pic}(\mathcal{C})$, the Picard group of $\mathcal{C}$. In this way $\mathrm{Pic}(\mathcal{C})$ appears as the parameter space for nonequivalent vacua. Our analysis is modelled on, but not limited to, the case when $\mathcal{C}$ is complex projective space $\mathbf{CP}^n$.

1. Introduction

Fibre bundles are powerful tools to formulate the gauge theories of fundamental interactions and gravity.[1] The question arises whether or not quantum mechanics may also be formulated fibre bundles.[a] Important physical motivations call for such a formulation.

In quantum mechanics one aims at constructing a Hilbert-space vector bundle over classical phase space. In geometric quantisation this goal is achieved in a two-step process that can be very succinctly summarised as follows. One first constructs a certain holomorphic line bundle (the *quantum line bundle*) over classical phase space. Next one identifies certain sections of this line bundle as defining the Hilbert space of quantum states. Alternatively one may skip the quantum line bundle and consider the one-step process of directly constructing a Hilbert-space vector bundle over classical phase space. Associated with this vector bundle there is a principal bundle whose fibre is the unitary group of Hilbert space.

Standard presentations of quantum mechanics usually deal with the case when this Hilbert-space vector bundle is trivial. Such is the case, e.g., when classical phase

*This research has been partially supported by DGICYT grant PB/6/FS/97.
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[a]The powerful tools of the gauge theories.

space is contractible to a point. However, it seems natural to consider the case of a nontrivial bundle as well. Beyond a purely mathematical interest, important physical issues that go by the generic name of *dualities*[2] motivate the study of nontrivial bundles.

Given a certain base manifold and a certain fibre, the trivial bundle over the given base with the given fibre is unique. This may mislead one to conclude that quantisation is also unique, or independent of the observer on classical phase space. In fact the notion of duality points precisely to the opposite conclusion, i.e. to the nonuniqueness of the quantisation procedure and to its dependence on the observer.[2]

Clearly a framework is required in order to accommodate dualities within quantum mechanics.[2] Nontrivial Hilbert-space vector bundles over classical phase space provide one such framework. They allow for the possibility of having different, nonequivalent quantisations, as opposed to the uniqueness of the trivial bundle.[b]

However, although nontriviality is a necessary condition, it is by no means sufficient. A flat connection on a nontrivial bundle would still allow, by parallel transport, to canonically identify the Hilbert-space fibres above different points on classical phase space. This identification would depend only on the homotopy class of the curve joining the basepoints, but not on the curve itself. Now flat connections are characterised by *constant* transition functions,[3] this constant being always the identity in the case of the trivial bundle. Hence, in order to accommodate dualities, we will be looking for *nonflat* connections.

First, we want to obtain the wave functions of a generalized pendulum under time-dependent gravitation by making use of a unitary transformation and the LR invariant method. As an example, we consider a generalized pendulum with exponentially increasing mass and constant gravitation. Second, we want to present a canonical approach for the generalized time-dependent pendulum which is based on the use of a time-dependent canonical transformation and an auxiliary transformation.

2. Properties of $\mathbf{CP}^n$ as a Classical Phase Space

We will consider a classical mechanics whose phase space $\mathcal{C}$ is complex, projective n-dimensional space $\mathbf{CP}^n$. The following properties are well known.[3]

Let $Z^1, \ldots, Z^{n+1}$ denote homogeneous coordinates on $\mathbf{CP}^n$. The chart defined by $Z^k \neq 0$ covers one copy of the open set $\mathcal{U}_k = \mathbf{C}^n$. On the latter we have the holomorphic coordinates $z^j_{(k)} = Z^j/Z^k$, $j \neq k$; there are $n+1$ such coordinate charts. $\mathbf{CP}^n$ is a Kähler manifold with respect to the Fubini-Study metric. On the chart $(\mathcal{U}_k, z_{(k)})$ the Kähler potential reads

$$K(z^j_{(k)}, \bar{z}^j_{(k)}) = \log\left(1 + \sum_{j=1}^{n} z^j_{(k)} \bar{z}^j_{(k)}\right) . \tag{1}$$

[b]The framework of the vector bundles.

The singular homology ring $H_*(\mathbf{CP}^n, \mathbb{Z})$ contains the nonzero subgroups

$$H_{2k}(\mathbf{CP}^n, \mathbb{Z}) = \mathbb{Z}\,, \qquad k = 0, 1, \ldots, n\,, \tag{2}$$

while

$$H_{2k+1}(\mathbf{CP}^n, \mathbb{Z}) = 0\,, \qquad k = 0, 1, \ldots, n-1\,. \tag{3}$$

We have $\mathbf{CP}^n = \mathbf{C}^n \cup \mathbf{CP}^{n-1}$, with $\mathbf{CP}^{n-1}$ a hyperplane at infinity. Topologically, $\mathbf{CP}^n$ is obtained by attaching a (real) $2n$-dimensional cell to $\mathbf{CP}^{n-1}$. $\mathbf{CP}^n$ is simply connected,

$$\pi_1(\mathbf{CP}^n) = 0\,, \tag{4}$$

it is compact, and inherits its complex structure from that on $\mathbf{C}^{n+1}$. It can be regarded as the Grassmannian manifold

$$\mathbf{CP}^n = \mathrm{U}(n+1)/(\mathrm{U}(n) \times \mathrm{U}(1)) = S^{2n+1}/\mathrm{U}(1)\,. \tag{5}$$

Let τ^{-1} denote the *tautological bundle* on $\mathbf{CP}^n$. We recall that τ^{-1} is defined as the subbundle of the trivial bundle $\mathbf{CP}^n \times \mathbf{C}^{n+1}$ whose fibre at $p \in \mathbf{CP}^n$ is the line in $\mathbf{C}^{n+1}$ represented by p. Then τ^{-1} is a holomorphic line bundle over $\mathbf{CP}^n$. Its dual, denoted τ, is called the *hyperplane bundle*. For any $l \in \mathbb{Z}$, the lth power τ^l is also a holomorphic line bundle over $\mathbf{CP}^n$. In fact every holomorphic line bundle L over $\mathbf{CP}^n$ is isomorphic to τ^l for some $l \in \mathbb{Z}$; this integer is the first Chern class of L.

2.1. *Computation of* $\dim H^0(\mathbf{CP}^n, \mathcal{O}(1))$

Next we present a quantum-mechanical computation of $\dim H^0(\mathbf{CP}^n, \mathcal{O}(1))$ without resorting to sheaf cohomology. That is, we compute $\dim \mathcal{H}$ when $l = 1$ and prove that it coincides with the right-hand side.

Starting with $\mathcal{C} = \mathbf{CP}^0$, i.e. a point p as classical phase space, the space of quantum rays must also reduce to a point. Then the corresponding Hilbert space is $\mathcal{H}_1 = \mathbf{C}$. The only state in $\mathcal{H}_1$ is the vacuum $|0\rangle_{l=1}$. Henceforth, for brevity, we drop the Picard class index from the vacuum.

2.2. *Representations*

The $(n+1)$-dimensional Hilbert space may be regarded as a kind of *defining representation*, in the sense of the representation theory of $\mathrm{SU}(n+1)$ when $n > 1$. To make this statement more precise we observe that one can replace unitary groups with special unitary groups in Eq. (5). Comparing our results with those of Sec. 2 we conclude that the quantum line bundle $\mathcal{L}$ now equals τ,

$$\mathcal{L} = \tau\,, \tag{6}$$

because $l = 1$. This is the smallest value of l that produces a nontrivial $\mathcal{H}$, gives a one-dimensional Hilbert space when $l = 0$. So our $\mathcal{H}$ spans an $(n+1)$-dimensional representation of SU$(n+1)$, that we can identify with the defining representation. There is some ambiguity here since the dual of the defining representation of SU$(n+1)$ is also $(n+1)$-dimensional. This ambiguity is resolved by convening that the latter is generated by the holomorphic sections of the *dual* quantum line bundle

$$\mathcal{L}^* = \tau^{-1} \,. \tag{7}$$

On the chart $\mathcal{U}_j$, $j = 1, \ldots, n+1$, the dual of the defining representation is the linear span of the covectors

$$\langle (j)0| \,, \qquad \langle (j)0|A_i(j) \,, \qquad i = 1, 2, \ldots, n \,. \tag{8}$$

Taking higher representations is equivalent to considering the principal SU$(n+1)$-bundle (associated with the vector $\mathbf{C}^{n+1}$-bundle) in a representation higher than the defining one. We will see next that this corresponds to having $l > 1$ in our choice of the line bundle τ^l.

3. Tangent Vectors as Quantum States

The converse is not true, as exemplified by the vacuum. Let us generalise and replace $\mathbf{CP}^n$ with an arbitrary classical phase space $\mathcal{C}$. We would like to write,

$$\mathcal{QH}(\mathcal{C}) = T(\mathcal{C}) \oplus N(\mathcal{C}) \,, \tag{9}$$

where $N(\mathcal{C})$ is a holomorphic line bundle on $\mathcal{C}$, whose fibre is generated by the vacuum state, and $T(\mathcal{C})$ is the holomorphic tangent bundle. Does Eq. (9) hold in general?

Table 1. This table gives the QES condition and the number of moving poles of χ for each combination of b_1 and b_1' for the Khare–Mandal model.

Set	$b1$ (Rad/s)	$b1'$ (Rad/s)	$n = \lambda_1 - b_1 - b_1'$	Condition o M	QES Condition
1	1/4	1/4	$\frac{M}{2} - \frac{1}{2}$	$M = \text{odd},\ M \geq 1$	$M = 2n+1$
3	3/4	1/4	$\frac{M}{2} - 1$	$M = \text{even},\ M \geq 2$	$M = 2n+1$
4	1/4	3/4	$\frac{M}{2} - 1$	$M = \text{even},\ M \geq 2$	$M = 2n+1$

The answer is also affirmative provided that $\mathcal{C}$ is a complex n-dimensional, compact, symplectic manifold, whose complex and symplectic structures are compatible. Notice that $\mathcal{C}$ is not required to be Kähler; examples of Hermitian but non-Kähler spaces are Hopf manifolds.[3] Let ω denote the symplectic form. Then $\int_{\mathcal{C}} \omega^n < \infty$ thanks to compactness,

$$\int_{\mathcal{C}} \omega^n = n + 1 \,. \tag{10}$$

Let us cover $\mathcal{C}$ with a *finite* set of holomorphic coordinate charts $(\mathcal{W}_k, w_{(k)})$, $k = 1, \ldots, r$; the existence of such an atlas follows from the compactness of $\mathcal{C}$. We can pick an atlas such that r is minimal; compactness implies that $r \geq 2$.

From Table 1, we see that sets 1 and 2 are valid only when M is odd and sets 3 and 4 are valid only when M is even.

4. Discussion

Quantum mechanics is defined on a Hilbert space of states whose construction usually assumes a global character on classical phase space. Under *globality* we understand, as explained in Sec. 1, the property that all coordinate charts on classical phase space are quantised in the same way.

A novelty of our approach is the local character of the Hilbert space: there is one on top of each Darboux coordinate chart on classical phase space. The patching together of these Hilbert-space fibres on top of each chart may be global (trivial bundle) or local (nontrivial bundle). In order to implement duality transformations we need a nonflat bundle (hence nontrivial). Flatness would allow for a canonical identification, by means of parallel transport, of the quantum states belonging to different fibres.

A duality thus arises as the possibility of having two or more, apparently different, quantum-mechanical descriptions of the same physics. Mathematically, a duality arises as a nonflat, quantum Hilbert-space bundle over classical phase space. This notion implies that the concept of a quantum is not absolute, but relative to the quantum theory used to measure it.[2] That is, duality expresses the relativity of the concept of a quantum. In particular, *classical* and *quantum*, for long known to be deeply related[11] are not necessarily always the same for all observers on phase space.

Acknowledgments

It is a great pleasure to thank J. de Azcárraga for encouragement and support. Technical discussions with U. Bruzzo and M. Schlichenmaier are gratefully acknowledged. The author thanks S. Theisen and Max-Planck-Institut für Gravitationsphysik,-Albert-Einstein-Institut, for hospitality. This work has been partially supported by research grant BFM2002-03681 from Ministerio de Ciencia y Tecnología, by EU FEDER funds, by Fundación Marina Bueno and by Deutsche Forschungsgemeinschaft.

Appendix A. Appendix

We can insert an Appendix here and includes equations which are numbered as Eq. (A.1),

$$\frac{4\pi}{3} r_{ij}^3 \cdot \frac{4\pi}{3} p_{ij}^3 = \frac{h^3}{4}\,. \tag{A.1}$$

Appendix A.1. *Subsection of Appendix*

$$\frac{5\pi}{10} r_{ij}^2 \cdot \frac{5\pi}{10} p_{ij}^7 = \frac{h^3}{4}\,. \tag{A.2}$$

The answer is trivially affirmative when $\mathcal{C}$ is an analytic submanifold of $\mathbf{CP}^n$. Such is the case, e.g., of the embedding of $\mathbf{CP}^n$ within $\mathbf{CP}^{n+l}$, Grassmann manifolds provide another example.

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