Geodesic Deviation in Kaluza Klein Scenario

Francesco Vietri ,Valentino Lacquaniti, Giovanni Montani

July 17, 2008

0.1 Geodesic Deviation with eletromagnetic field

Start from the motion of a particle in an electromagnetic field in the scenario of General Relativity :

$$\frac{DU^{\alpha}}{Ds} = \frac{q}{m} F^{\alpha\beta} \, U_{\beta}$$

Now we imagine two geodetic very close between them in a elettromagnetic field: the geodetic deviation with the Elettromagnetic field will be the difference between

$$\frac{DU^{\alpha}}{Ds} = \frac{q}{m} F^{\alpha\beta} \, U_{\beta}$$

and

$$\frac{D^2\left(X^{\alpha} + \delta x^{\alpha}\right)}{Ds^2} = \frac{q}{m} F\left(X^{\mu} + \delta x^{\mu}\right)^{\alpha\ \beta} \, \frac{D^2\left(X_{\beta} + \delta x_{\beta}\right)}{Ds^2}$$

with the infinitesimal displacement δx^{α}

and so we have :

$$\frac{D^2 \delta x^{\alpha}}{Ds^2} = -R^{\alpha}_{\beta\gamma\lambda} U^{\beta} \delta x^{\gamma} U^{\lambda} + \frac{q}{m} \delta x^{\nu} \nabla_{\nu} \left[F^{\alpha\beta} \, U_{\beta}\right]$$

0.2 Geodesic equation in Kaluza Klein Theories

In the scenario of Kaluza Klein we calculate the geodetic using the variational principle:

$$\frac{DU^{a}}{DS} = 0$$

after the geometrical reduction of the equation we can build the correct motion equation:

$$\frac{DU^{\gamma}}{DS} = \frac{U_5}{\sqrt{1 + \frac{U_5^2}{\Phi 2}}} F^{\gamma\beta} \, U_{\beta} \ + \frac{1}{\Phi^3} \left(U^{\gamma} U^{\beta} - g^{\gamma\beta}\right) \partial_{\beta} \Phi \left(\frac{U_5^2}{1 + \frac{U_5^2}{\Phi^2}}\right)$$

while the fifht component give us a law of conservation along the direction of motion

$$\frac{DU_5}{DS} = 0$$

now if we take the particolar condition $\Phi = 1$:

$$\frac{DU^{\alpha}}{Ds} = \frac{U_5}{\sqrt{1+U_5^2}} F^{\alpha\beta}\ U_{\beta}$$

and we define as ratio q over m

$$\frac{U_5}{\sqrt{1+U_5^2}} = \frac{q}{m} \leq 1$$

We can rebuild the correct geodosic equation with the eletromagnetic field.

We want to point out that the problem of the correct ratio q over m arises from the reparametrization factor from ds_5 to ds

$\omega = \sqrt{1+U_5^2/\Phi^2} = \frac{ds}{ds_5}$

this factor generates the problem of the upper bound in q/m and the non conservation of q/m if "$\Phi \neq 1$".

0.3 Geodesic Deviation in Kaluza Klein Theories

Now we will take two 5-D equations geodesica very close between them and we operate as the 4-D equations geodesica to find the Geodesic Deviation and immediately we get the geometrical reduction and investigate the case $\Phi = 1$

$$\frac{D^2\delta x^{\alpha}}{Ds^2} = -R^{\alpha}_{\beta\gamma\lambda}U^{\beta}\delta x^{\gamma}U^{\lambda} + \frac{U_5}{\sqrt{1+U_5^2}}\delta x^{\nu}\nabla_{\nu}\left[F^{\alpha\beta}U_{\beta}\right] - \frac{U_5}{\sqrt{1+U_5^2}}F^{\alpha}_{\nu}U^{\nu}\left(\frac{d\delta x_5}{ds} - F_{\mu\nu}\delta x^{\mu}U^{\nu}\right)$$

for the fifth component we have

$$\frac{d}{ds}\left(\frac{d\delta x_5}{ds} - F_{\mu\nu}\delta x^{\mu}U^{\nu}\right) = 0$$

this means :

$$\frac{D\delta x_5}{Ds} - F_{\mu\nu}\delta x^{\mu}U^{\nu}\ \ = \delta Q = constant$$

this constant represents the difference of charge between two charged particles very close moving on two different geodesic path

$$\frac{D^2\delta x^{\alpha}}{Ds^2} = -R^{\alpha}_{\beta\gamma\lambda}U^{\beta}\delta x^{\gamma}U^{\lambda} + \frac{U_5}{\sqrt{1+U_5}}\delta x^{\nu}\nabla_{\nu}\left[F^{\alpha\beta}U_{\beta}\right] - \frac{U_5}{\sqrt{1+U_5}}F^{\alpha}_{\nu}U^{\nu}\delta Q$$

defining : "'$\delta Q = 0$"

$$\frac{U_5}{\sqrt{1+U_5^2}} = \frac{q}{m}$$

we retrieve the equation 4-D departure

Now we investigate the case $\Phi \neq 1$:

$$\frac{D^2 \delta x^\alpha}{Ds^2} = -R^\alpha_{\beta\gamma\lambda} U^\beta \delta x^\gamma U^\lambda + \frac{U_5}{\omega} \delta x^\nu \nabla_\nu \left[F^{\alpha\beta} U_\beta \right] +$$

$\delta x^\nu \nabla_\nu \left(\frac{\partial^\alpha \Phi}{\Phi^3} \right) \frac{U_5^2}{\omega^2} -$

$- \frac{U_5}{w} \left(F^\alpha_\nu U^\nu - 2 \frac{U_5}{\omega} \frac{\partial^\alpha \Phi}{\Phi} \right) \left(\frac{d\delta x_5}{ds} - 2 \frac{\delta x_5}{\Phi} \frac{d\Phi}{ds} - \Phi^2 F_{\mu\nu} \delta x^\mu U^\nu \ + 2 \frac{U_5}{\omega} \frac{\delta x^\rho \partial_\rho \Phi}{\Phi} \right)$

Now we investigate the fifth component of the equation:

$$\frac{d}{ds} \left[\frac{U_5}{\omega} \left(\frac{d\delta x_5}{ds} - 2 \frac{\delta x_5}{\Phi} \frac{d\Phi}{ds} - \Phi^2 F_{\mu\nu} \delta x^\mu U^\nu \ + 2 \frac{U_5}{\omega} \frac{\delta x^\rho \partial_\rho \Phi}{\Phi} \right) \right] = -\frac{1}{\omega} / \Phi \frac{\delta x^\nu \partial_\nu \Phi}{\Phi^2} \frac{d\Phi}{ds}$$

As we can see in this case, we do not have the conservation of the difference of charge, as we have seen before. The difference of charge is linked a to a function of Φ; this problem arises from the reparametrization parameter.
Now we espress the fuction that represents the variation of the difference of charge in a difference way:

$$\frac{d\omega}{ds} = -\frac{1}{\omega^2} \frac{d\Phi}{ds}$$

$\delta \left(\omega \right) = \frac{U_5^2}{\omega^2} \left(\frac{U_5}{\Phi^2} \frac{d\delta_5}{ds} - \frac{U^5}{\Phi^2} \delta x^\nu \partial_\nu \Phi \right)$

$\frac{1}{\omega^2} \frac{\delta x^\nu \partial_\nu}{\Phi^2} \frac{d\Phi}{ds} = \left[\left(\omega^2 \right) \delta \left(\sqrt{\omega} \right) \frac{\Phi^3}{U_5^2} - \sqrt{\omega} \frac{\Phi}{U_5} \frac{\delta x_5}{ds} \right] \left(\frac{d\sqrt{\omega}}{ds} \right)$

Now with an obviusly generalizzation we define the difference of charge as $\delta Q = \frac{U_5}{\omega} \left(\frac{d\delta x_5}{ds} \frac{\delta x_5}{\Phi} \frac{d\Phi}{ds} - \Phi^2_{\mu\nu} \delta x^\mu U^\nu + 2 \frac{U_5}{\omega} \frac{\delta x^\rho \partial_\rho \Phi}{\Phi} \right)$

$$\frac{d}{ds} \delta Q = \left[\omega^2 \delta \left(\omega \right) \frac{\Phi^3}{U_5^2} - \omega \frac{\Phi}{U_5} \frac{\delta x_5}{ds} \right] \left(\frac{d\omega}{ds} \right)$$

0.4 conclusion

1. We know that for the simple case ,

$$"\phi = 1"$$

we find an equation very similar to the equation of the 4-D scenario with a condition that represent the conservation of difference of charge during the motion, this condition is a starting condition of motion we are free to choose every value in the idea to restore the 4-d equation we have to put this condition equal to zero.

2. In the general case,

$$"`\phi \neq 1"$$

we have lost the conservation of difference of charge and we can see that this problem is linked with problem of the definition of the displacement factor between ds five and ds , this problem of the definition is orginated from the definiton of the equation of motion in Kaluza Klein scenario using the variational principle and the broken of the equivalence principle in 5-D.