Electric force lines and Stability in the Alekseev-Belinski solution

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What is the problem

1. The equilibrium of two charged masses:

   - In newtonian physics: $m_1 m_2 = e_1 e_2$
   - In GR one must solve the Einstein-Maxwell system:

   $$\begin{cases} R_{ij} - \frac{1}{2}R_{ij} = \frac{1}{2}\left(-F_{ik}F^{k}{}_{j} + \frac{1}{4}F_{lm}F^{lm}g_{ij}\right) \\ (\sqrt{-g}F^{ik})_{,k} = \sqrt{-g}j^{i} \end{cases}$$

   and find a solution **without conic singularities**

   The main differences between classic and relativistic regime arise from the repulsive nature of gravity near the charge. That can be seen also by the RN metric

   $$g_{tt} = 1 - \frac{2M}{r} + \frac{Q^2}{r^2},$$

   where gravity is repulsive for

   $$r < \frac{Q^2}{M}$$

2. The gravitational and electric fields generated.

Some historical remarks

Perturbation Methods	Exact Solutions
Copson (1927) **Electric field of a test charge near a Schwarzschild b.h.**	
	$Majumdar - Papapetrou$ (1947) $m_i = e_i$
$Hanni - Ruffini$ (1973) **Electric force lines of a test charge near a Schwarzschild b.h.**	
Linet (1976) **A correction of Copson solution**	
	$Belinski - Zakharov$ (1978) **Vacuum Solitons** and *Alekseev* (1980) **Electrovacuum Solitons** Solutions of $Hauser - Ernst$ (1979) *and Sibgatulling* (1984) by IEM for rational axis data, and of *Alekseev*(1985) by IEM for rational monodromy data **Integral Equation Method**
Bonnor (1993) **Equilibrium of a test particle on RN background**	
$Perry - Cooperstock$ (1997) **Equilibrium is possible** (3 numerical examples)	
$Bini - Geralico - Ruffini$ (2007) **Equilibrium of a test charge on RN with back-reaction until first order**	$Alekseev - Belinski$ (2007) **Exact solution for equilibrium (without strut) of two RN sources**

The Alekseev-Belinski Solution

<<`Simplex sigillum veri`>>

The explicit form of the Alekseev-Belinski solution is:

$$ds^2 = H(\rho, z)dt^2 - \frac{\rho^2}{H(\rho, z)}d\varphi^2 - f(\rho, z)(d\rho^2 + dz^2) \tag{1}$$

$$A_t = \Phi(\rho, z), \ A_\varphi = A_\rho = A_z = 0, \tag{2}$$

with

$$H = \frac{[(r_1 - m_1)^2 - \sigma_1^2 + \gamma^2 \sin^2\theta_2][(r_2 - m_2)^2 - \sigma_2^2 + \gamma^2 \sin^2\theta_1]}{D^2} \tag{3}$$

$$\Phi = \frac{[(e_1 - \gamma)(r_2 - m_2) + (e_2 + \gamma)(r_1 - m_1) + \gamma(m_1 \cos\theta + m_2 \cos\theta_2)]}{D} \tag{4}$$

$$f = \frac{D^2}{[(r_1 - m_1)^2 - \sigma_1^2 \cos^2\theta][(r_2 - m_2)^2 - \sigma_2^2 \cos^2\theta_2]}$$

where

$$D = r_1 r_2 - (e_1 - \gamma - \gamma\cos\theta_2)(e_2 + \gamma - \gamma\cos\theta), \tag{5}$$

while γ, σ_1 and σ_2 are defined by:

$$\begin{gathered} \gamma = (m_2 e_1 - m_1 e_2)(l + m_1 + m_2)^{-1}\,, \\ \sigma_1^2 = m_1^2 - e_1^2 + 2e_1\gamma\,, \qquad \sigma_2^2 = m_2^2 - e_2^2 - 2e_2\gamma. \end{gathered} \tag{6}$$

The mathematical parameters coincide with the physical ones, therefore $M_{tot} = m_1 + m_2$ and $Q_{tot} = e_1 + e_2$.

The above formulas give the solution of the Einstein-Maxwell system only if l satisfies the equilibrium condition

$$m_1 m_2 = (e_1 - \gamma)(e_2 + \gamma). \tag{7}$$

The distance l can be written as a function of the other parameters by the very simple formula:

$$l = -m_1 - m_2 + \frac{m_1 e_2 - m_2 e_1}{2(m_1 m_2 - e_1 e_2)} \left[(e_2 - e_1) \pm \sqrt{(e_1 + e_2)^2 - 4\, m_1 m_2} \right].$$

By definition $l \equiv z_2 - z_1$ is the distance, expressed in the Weyl coordinate z, between the two objects.

From (33) is clear that the parameters must satisfy the restriction

$$(e_2 + e_1)^2 > 4\, m_1 m_2, \tag{8}$$

furthermore we impose the non-overlapping condition

$$l > \sigma_1. \tag{9}$$

In *Proc. of XI Marcel Grossman Meeting (Berlin, July 2006), arXiv:gr-qc/0710.2515, 2007.* Alekseev and Belinski gave also the solution for arbitrary l.

Possible/Not possible Configurations

1. `extreme-extreme source` (well known, Majumdar-Papapetrou);

2. extreme-not extreme: not allowed;

3. `black hole-black hole`: not possible without overlapping;

4. `naked-naked`: not possible at all;

5. **black hole-naked singularity**: possible in a certain range of m_1, e_1, m_2, e_2.

   Only in this case (and in case 1.) it exists the Newtonian limit (large l).

Electric force lines definition

We define the electrical vector by the time-like components of the controvariant tensor F^{ij}:

$$E^{\alpha} = F^{\alpha 0}. \tag{10}$$

That definition is justified by the Gauss theorem (Wheeler, Geometro-dynamics):

$$4\pi Q = \int_C {}^*\mathbf{F} = \int_C {}^*F_{ij} dx^i \wedge dx^j, \tag{11}$$

where ${}^*F_{ij} = 1/2\epsilon_{ijkl}F^{kl}\sqrt{-g}$.

Then it is natural to define (Hanni-Ruffini) the force lines as the solution of the differential system:

$$\begin{cases} \frac{d}{d\lambda} r_1 = E^{r_1} \\ \\ \frac{d}{d\lambda}\theta_1 = E^{\theta_1} \end{cases} \tag{12}$$

or equivalently by

$$\frac{dr_1}{d\theta_1} = \frac{E^{r_1}}{E^{\theta_1}}, \qquad \frac{E^{r_1}}{E^{\theta_1}} = ((r_1 - m_1)^2 - \sigma_1^2)\frac{\partial_{r_1}\Phi}{\partial_{\theta_1}\Phi}. \tag{13}$$

Physical interpretation (Christodoulou-Ruffini): a force line is a line tangent to the direction of the electric force measured by a free-falling test charge momentarily at rest, with initial 4-velocity

$$u^t = (\sqrt{g^{tt}}, 0, 0, 0). \tag{14}$$

Note: that interpretation is valid only for $g^{tt} > 0$.

A: Two RN with charge of the same sign

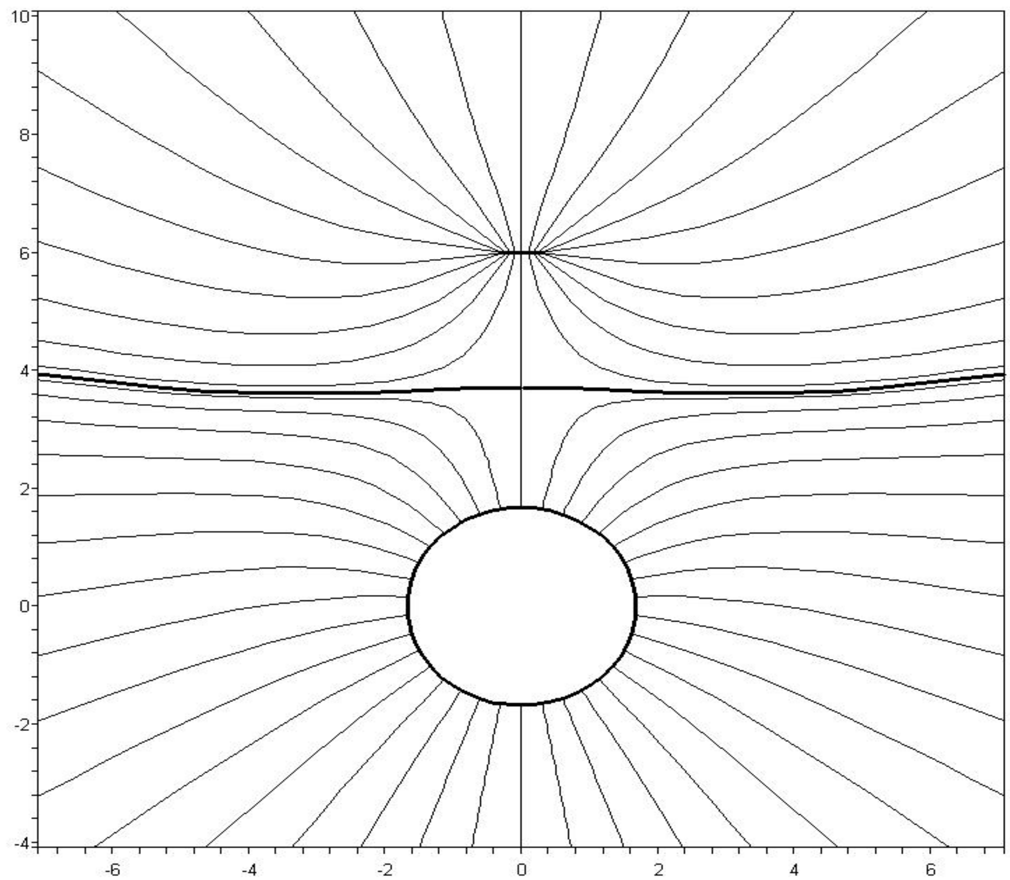

Figure 1: Force lines in a general case; the two RN have **charges of the same sign**. Note that the critical spheroid in that coordinate representation is an horizontal segment. Parameters used: $m_1 = 1$, $e_1 = 0.7$, $m_2 = 0.33$, $e_2 = 0.44$.

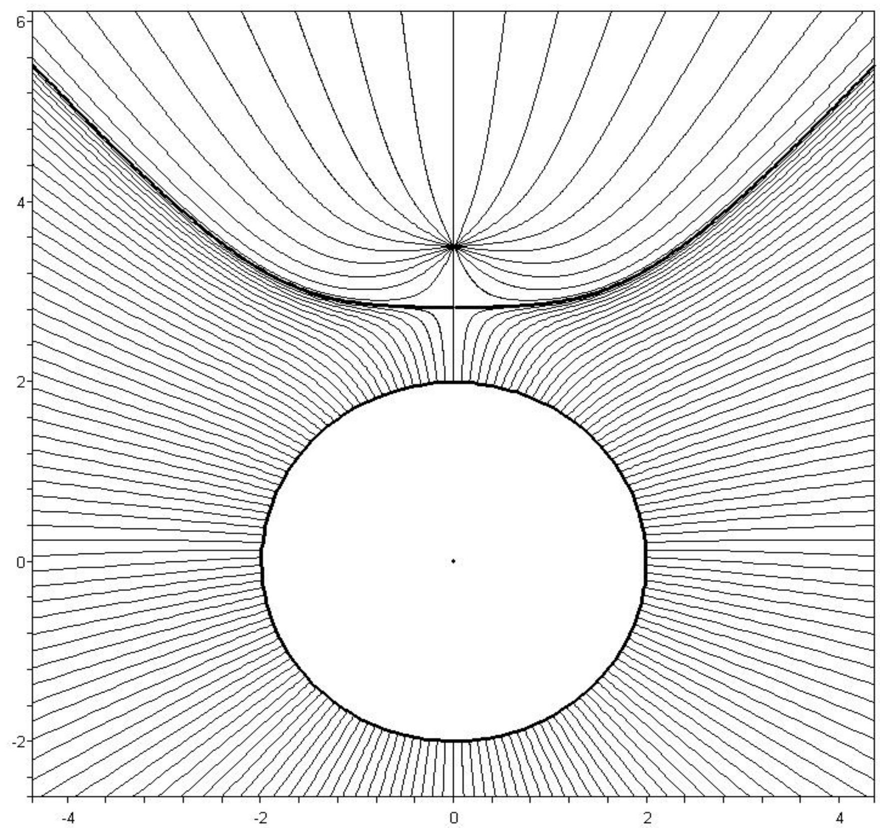

Figure 2: Force lines of a test charge near a RN with horizon. Parameters used: $m_1 = 1$, $e_1 = 0.1$, $m_2 = 10^{-3}$, $e_2 = 1.3 \cdot 10^{-2}$. **Unstable configuration.**). The bold line is the separatrix.

B: Two RN with charge of the opposite sign

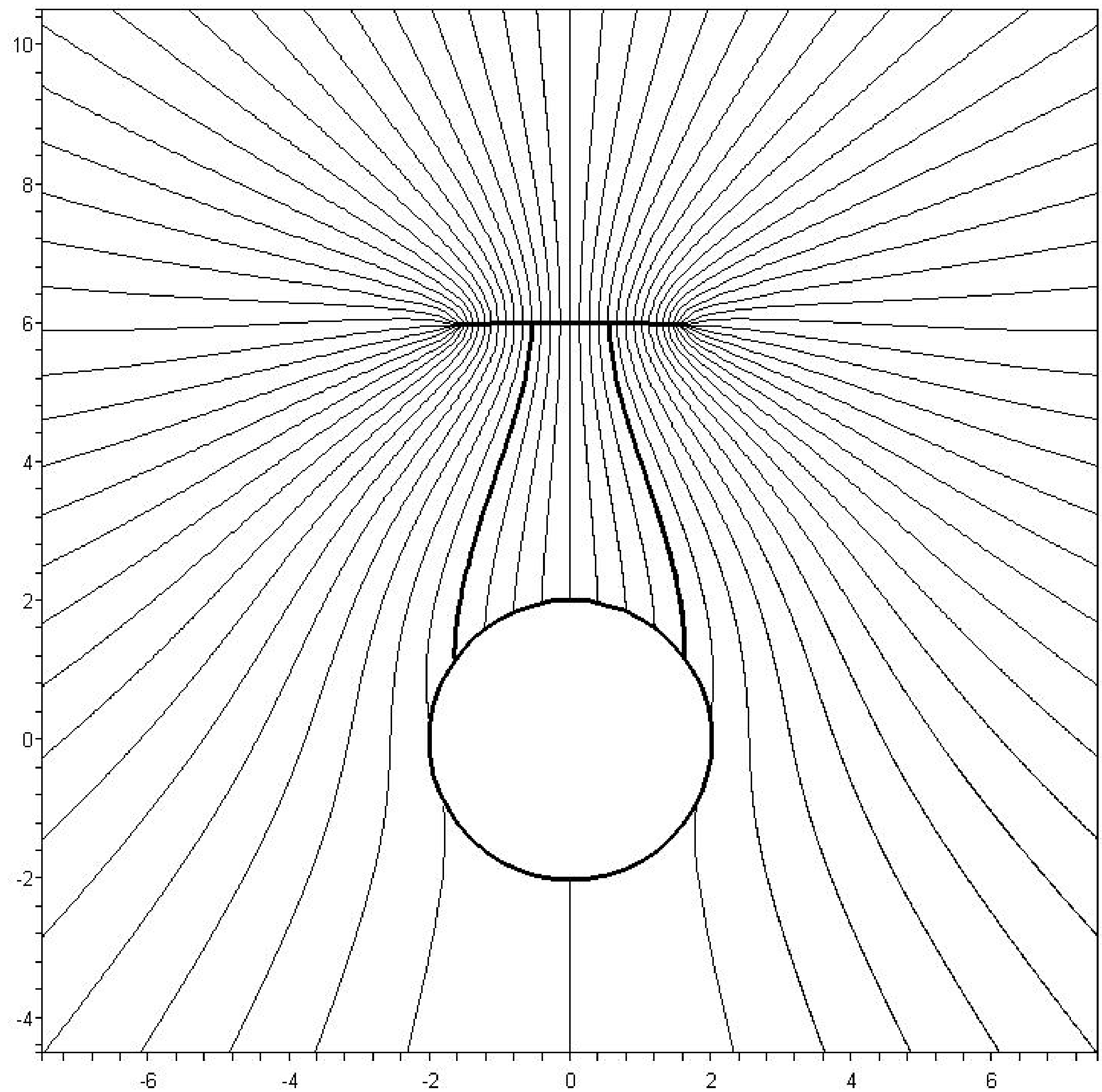

Figure 3: Force lines with **charges of the opposite sign**. Parameters used: $m_1 = 1$, $e_1 = 0.05$, $m_2 = 0.3$, $e_2 = -1.66$. The bold line is the separatrix, which now encircles also the central singularity of the b.h.: inside that region the lines go from one charge to the other. Outside that region the lines go from e_2 to infinity (some of them pass also through the horizon).

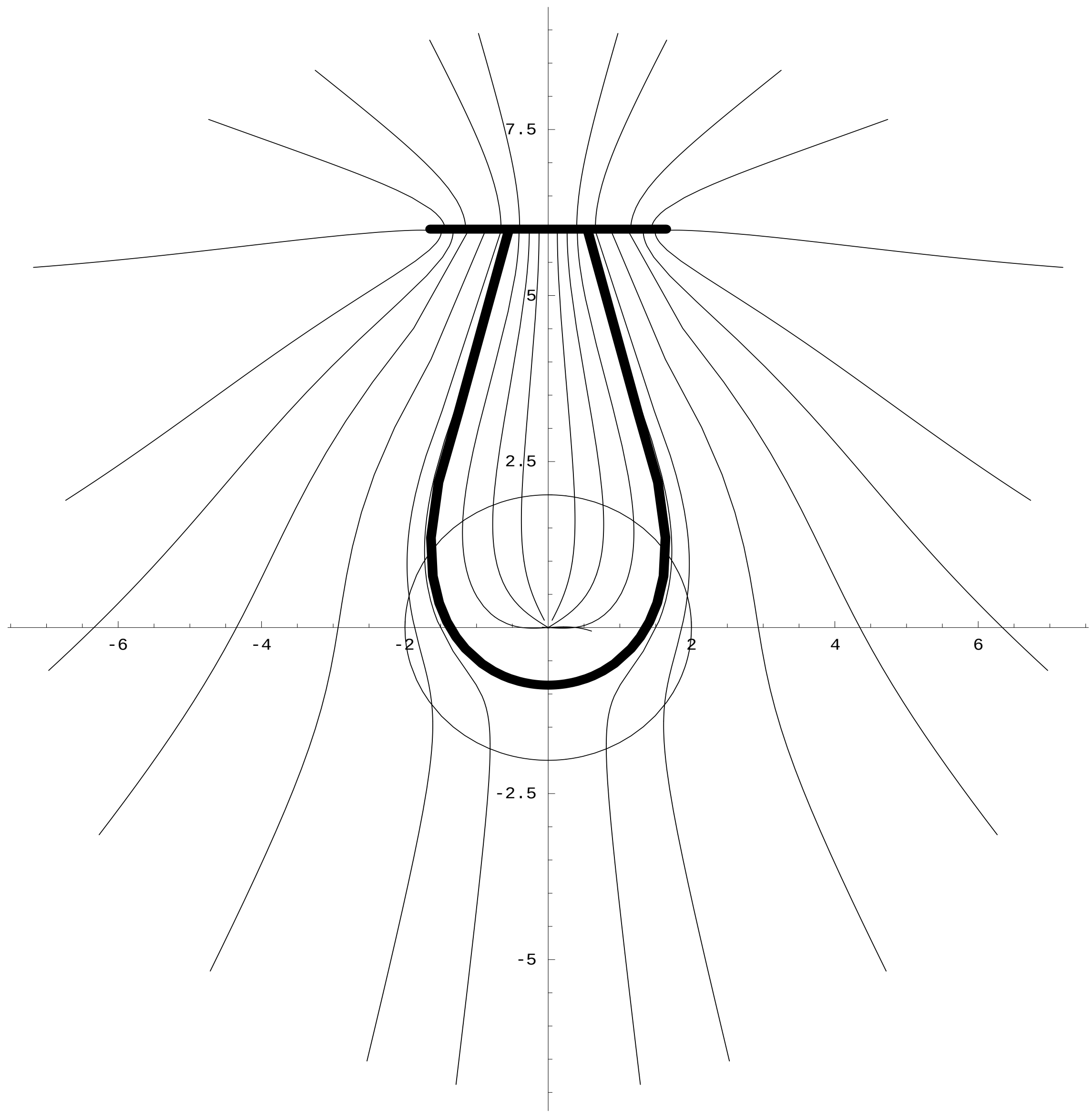

Figure 4: Mathematical continuation of the force lines inside the horizon. (**Warning**: Forbidden picture!—“uncensored” version of the previous one).

C: Cases with only one charge: RN near Schwarzschild

In the particular case in which the first source is neutral (i.e. $e_1 = 0$), the equilibrium distance is even simpler,

$$l = -m_1 - m_2 + \frac{e_2^2}{2m_2}\left(1 + \sqrt{1 - 2\,m_1\left(\frac{e_2^2}{2m_2}\right)^{-1}}\right), \tag{15}$$

which can be always satisfied for sufficiently large values of the charge parameter e_2.

It is worth noting that

$$\frac{e_2^2}{2m_2} > 2m_1 \tag{16}$$

the "geometrical size" of the naked singularity cannot be smaller than the Schwarzschild radius. Anyway it is possible to construct membrane-like models of a naked singularity with an external repulsive region. (V.Belinski, M.Pizzi, *Charged membrane as a source for repulsive gravity*, submitted to IJMPD)

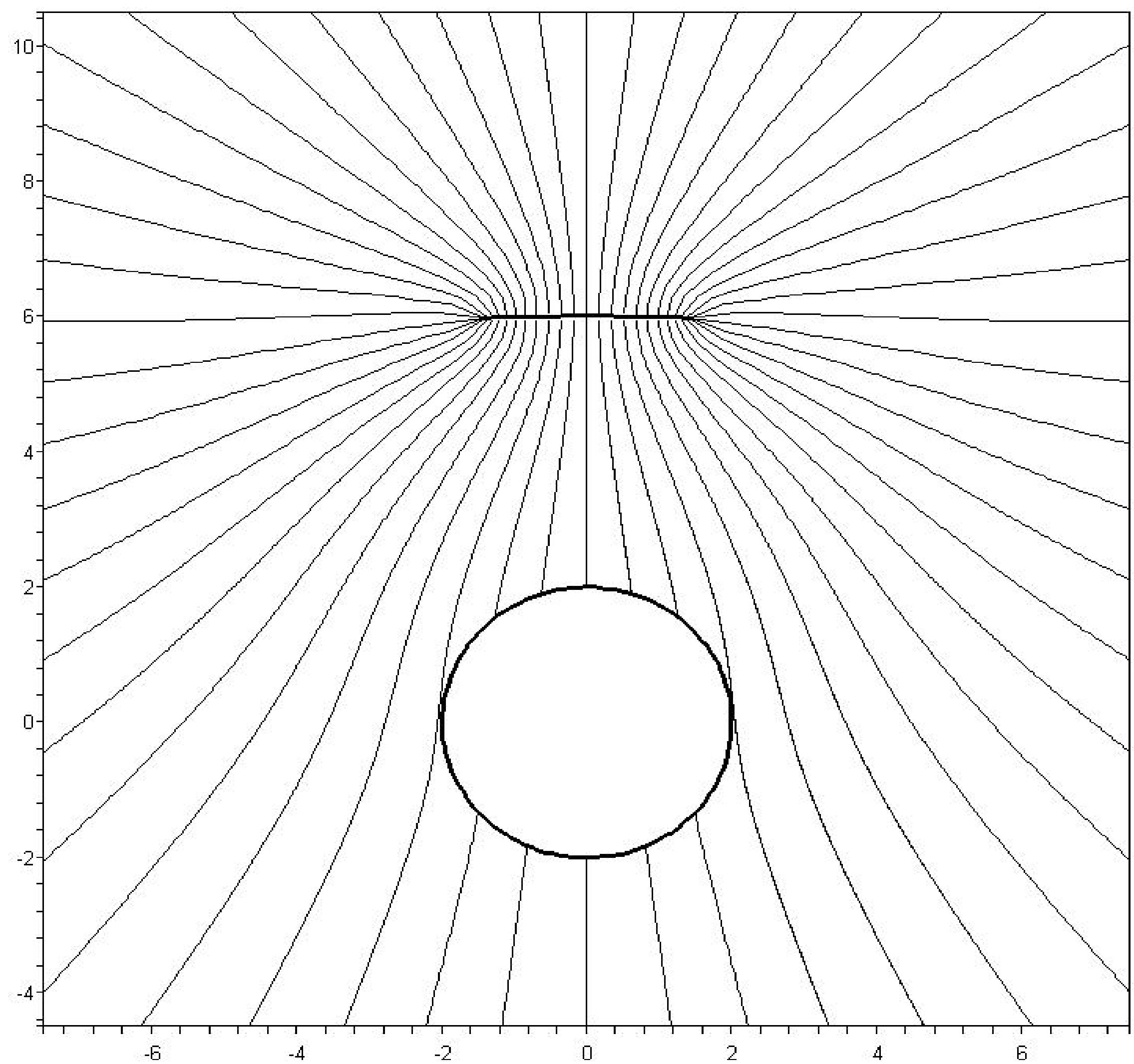

Figure 5: Force lines. The blank circle of radius $2m_1$ is the Schwarzschild horizon. Parameters used: $m_1 = 1$, $m_2 = 0.3$, $e_2 = 1.5$.

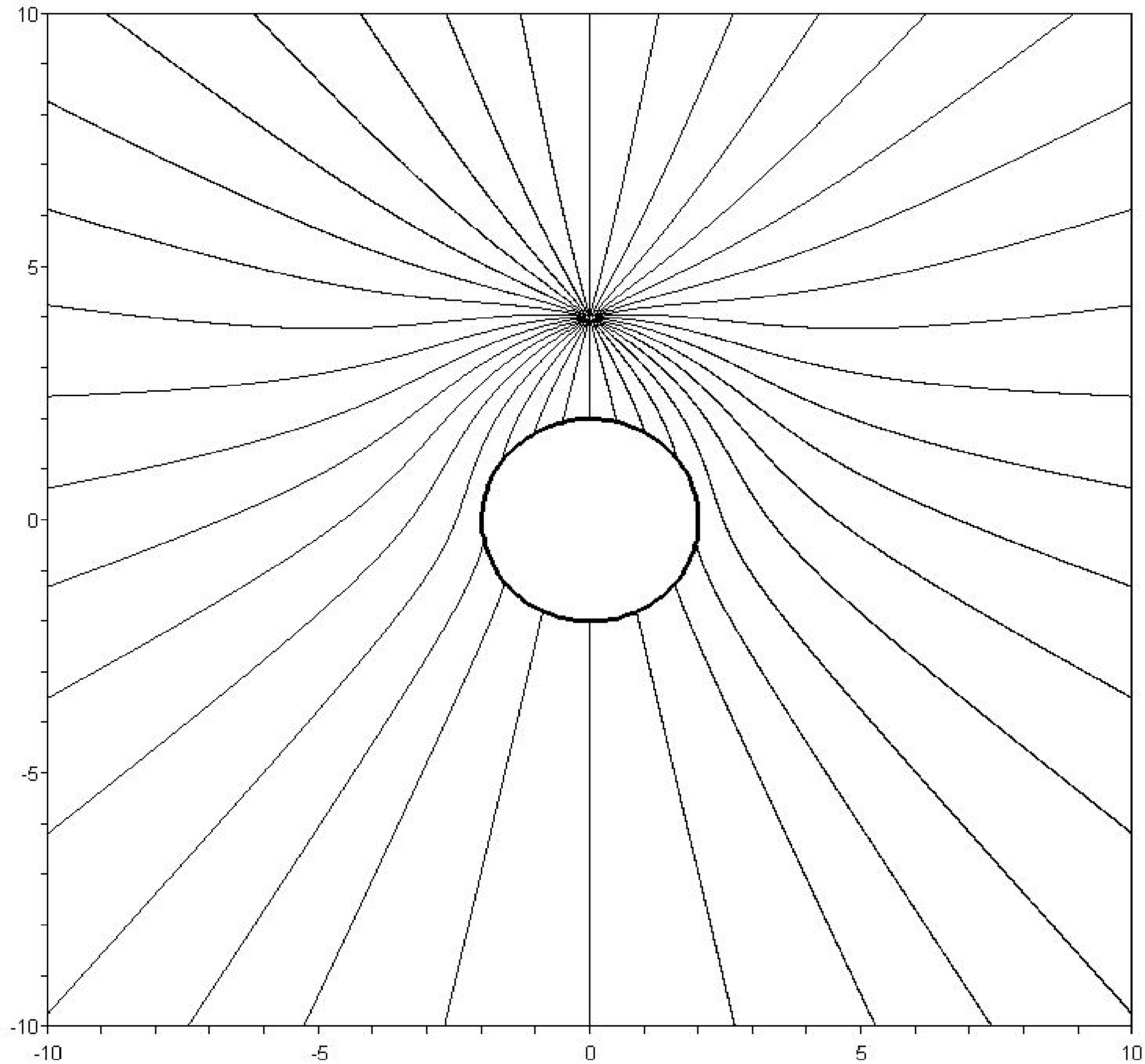

Figure 6: Force lines with $l = 3\,m_1$, with $m_1 = 1$, $m_2 = 10^{-4}$ and $e_2 = 0.023$. The circle of radius $2m_1$ is the Schwarzschild horizon. The plots are practically identical to the ones found by Hanni and Ruffini.

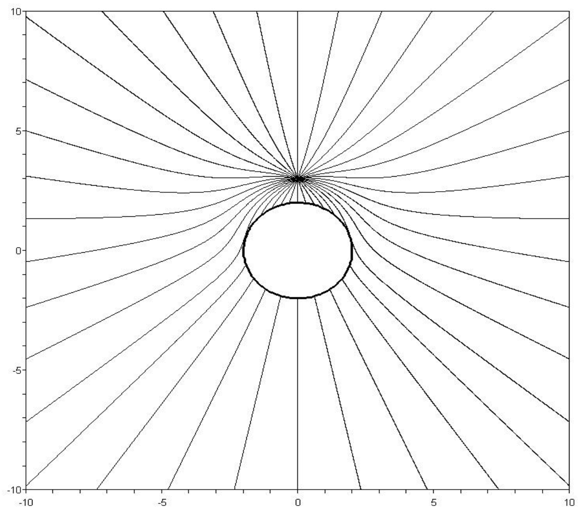

Figure 7: Now the distance is $l = 2\,m_1$, or equivalently $r_1 = 3\,m_1$.

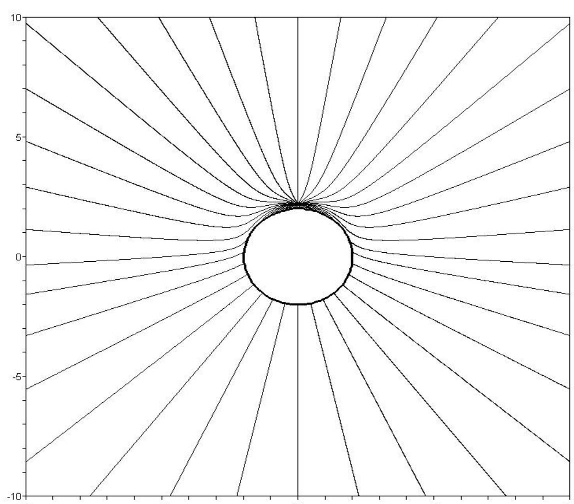

Figure 8: Now the distance is $l = 1.2\,m_1$, or equivalently $r_1 = 2.2\,m_1$.

Stability

`<<I am sorry for the longness of my letter, I had not the time to write a shorter one.>>`

(Pascal, Lettres Provinciales)

1. Perturbation of the metric with respect to the reciprocal distance between the two body

2. Strut's force definition (Sokolov and Starobinski)

3. Analysis of the AB solution in the three different cases:
   $e_1 e_2 > 0, \quad e_2 e_2 < 0, \quad e_1 = 0$

Reference: M.Pizzi, A. Paolino, *Stability in the Alekseev-Belinski solution*, to be published.

- When the solution is calculated for an arbitrary value of the distance $l = l_0 + x$ it appears a conic singularity between the two bodies (which is interpreted as a strut).

- Now, *we assume that in the reality there will be no struts if the two bodies will be displaced from the equilibrium position*, but that the force exercised by the two bodies one to the other will be precisely the opposite of F_{Strut}, say

$$F_{Bodies} = -F_{Strut} \; ; \tag{17}$$

indeed the eventual presence of a strut with such a force would balance the repulsion/attraction of the bodies, keeping the system exactly in "equilibrium", with $F_{tot} = F_{Bodies} + F_{Strut} = 0$.

- We want to calculate the energy-momentum tensor T_i^j on this segment in order to calculate the force of the strut. Since we know the metric it is convenient to define T_i^j by the Einstein equations $8\pi T_i^j = R_i^j - 1/2\delta_i^j R$.

- On the axis we can approximate the metric as:

$$ds^2 = d\tilde{t}^2 - d\tilde{z}^2 - (d\rho^2 + a^2\rho^2 d\varphi^2), \qquad a = \frac{1}{(\sqrt{fH})_{\rho=0}}. \tag{18}$$

$a \neq 1$ between the two sources. A direct calculation of R_i^j gives $R_i^j = 0$, $R = 0$, and thus $T_i^j = 0$.

- However we can introduce a distribution-like source using the Gauss-Bonnet theorem (D.D. Sokolov and A.A. Starobinski, *Sov. Phys. Dokl.*, 22(6), 312, 1977):

$$\int_S K d\sigma = 2\pi - \int_{\partial S} k_g ds \qquad \text{(Gauss-Bonnet th.)} \tag{19}$$

K is the Gaussian curvature, it is the half of the Ricci scalar, $K = 2R$; k_g is the geodesic curvature:

$$k_g = \epsilon_{ij}\left(\frac{d^2x^i}{ds}\frac{dx^j}{ds} + \Gamma^i_{kl}\frac{dx^k}{ds}\frac{dx^l}{ds}\frac{dx^j}{ds}\right)\left(g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\right)^{-1/2} \tag{20}$$

If S is a (small) disk of radius ϵ around the surface, we obtain:

$$R = \pi\frac{1-a}{a}\delta(\rho), \qquad \int_0^\epsilon\int_0^{2\pi}\rho\delta(\rho)d\rho d\varphi \equiv 1 \tag{21}$$

the result is independent of the radius of the disk.

- Then, using $R^z_z = R/2$ and the Einstein eqns. we find:

$$T^z_z = \frac{1-a}{4a}\delta(\rho), \tag{22}$$

using which we find the expression of the force:

$$\begin{aligned} F_{Strut} =& -\int_0^{\epsilon}\int_0^{2\pi} T^z_z \sqrt{g_{\rho\rho}g_{\varphi\varphi}} d\rho d\varphi \\ =& \frac{1}{2}(a-1) \end{aligned} \tag{23}$$

where

$$a = \left[1 - 2\frac{m_1 m_2 - (e_1-\gamma)(e_2+\gamma)}{(l_0+x)^2 - m_1^2 - m_2^2 + (e_1-\gamma)^2 + (e_2+\gamma)^2}\right]^{-1} \tag{24}$$

- In the large distance limit it gives Coulomb-Newtonian result

$$F_{Strut} = \frac{m_1 m_2 - e_1 e_2}{l_0^2}. \tag{25}$$

- Therefore the stability can be deduced from the sign of the derivative of the force w.r.t. x; obviously we have to evaluate this quantity on the equilibrium point $x = 0$:

$$(\partial_x F_{Bodies})_{x=0} = \frac{m_2 e_1 - m_1 e_2}{(l_0 + m_1 + m_2)^2}\left[\frac{2\gamma_0 - e_1 + e_2}{l_0^2 - m_1^2 - m_2^2 + (e_1-\gamma_0)^2 + (e_2+\gamma_0)^2}\right]; \tag{26}$$

where γ_0 is γ evaluated on $x = 0$; the stability condition is:

$$(\partial_x F_{Bodies})_{x=0} < 0. \tag{27}$$

- The previous formula can be simplified without loss of generality using the following considerations:

1. Using the arbitrariness of the electric charge's sign definition:

$$e_2 > 0; \tag{28}$$

2. Since we are considering a black hole and a naked singularity:

$$\frac{e_1}{m_1} < 1 < \frac{e_2}{m_2}; \tag{29}$$

3. Separability requirement:

$$l_0 > \sigma_1; \tag{30}$$

4. Finally, the existence of a real l_0 needs:

$$(e_2 + e_1)^2 > 4\, m_1 m_2. \tag{31}$$

- Using the previous conditions, the stability condition (27) can be reduced to the following one:

$$X \equiv (m_2 - m_1)(e_1 + e_2) + (e_2 - e_1) l_0 > 0; \tag{32}$$

X is an irrational 4-parameters quantity, remembering that

$$l_0 = -m_1 - m_2 + \frac{m_1 e_2 - m_2 e_1}{2(m_1 m_2 - e_1 e_2)} \left[(e_2 - e_1) \pm \sqrt{(e_1 + e_2)^2 - 4\, m_1 m_2} \right].$$

(A) Equal signed charges ($e_1 > 0$, $e_2 > 0$)

This is the presumably-only case in which we found also unstable equilibria.

Sub-case (A.1) $m_1 < m_2$. (BH smaller than NS)

If $m_2 > m_1$ then, necessarily from(29), $e_2 > e_1$: consequently $X > 0$ is always satisfied and the equilibrium is always stable.

Sub-case (A.2) $m_2 < m_1$. (BH larger than NS)

Sub-sub-case (A.2.1) $m_2 < m_1$ and $e_1 < e_2$.

Numerically we found only stable equilibrium (when it exist).

Sub-sub-case (A.2.2) $m_2 < m_1$ and $e_2 < e_1$. (Unstable)

In this case X is always negative and thus the equilibrium is unstable. A particular situation of this sub-case is the small-particle limit ($m_2 = \alpha e_2^2$, $e_2 \to 0$, $\alpha < 1$ constant), see e.g. plot (2) considered before.

That agrees with the instability found by **Bonnor** (*Class. Quantum Grav. 10, 2077-2082*, 1993).

(B) Opposite signed charges ($e_1 < 0$, $e_2 > 0$)

In this case we suspect that the equilibrium is always stable. Indeed, we was not able to found unstable equilibria, but we cannot assert that thy do not exist because there is one subcase in which we was not able to demonstrate analytically. Anyway, we can demonstrate that this is true in the following subcases.

Since X now is

$$X = (m_2 - m_1)(e_2 - |e_1|) + (e_2 + |e_1|)l_0, \tag{33}$$

then we can consider the two different sub-cases: $m_2 > m_1$ and $m_1 > m_2$.

Sub-case (B.1) $m_2 > m_1$. (BH smaller than NS)

If $m_2 > m_1$, then from condition (29) we have necessarily $e_2 > e_1$, which implies that X is always positive.

Sub-case (B.2) $m_1 > m_2$. (BH larger than NS)

Otherwise, if $m_1 > m_2$, then we need to consider the two different sub-sub-cases: $|e_1| \gtrless e_2$.

Sub-sub-case (B.2.1) $m_1 > m_2$, $|e_1| > e_2$.

If $|e_1| > e_2$ then one can see at first sight from (33) that X is always positive.

Sub-sub-case (B.2.1) $m_1 > m_2$, $|e_1| < e_2$.(Stable?)

If $|e_1| < e_2$ we are not able to demonstrate that X is always positive, but we can say that at least for enough large values of e_2 this is true, because

$$\lim_{e_2 \to \infty} l_0 \approx \frac{m_1}{|e_1|} e_2, \tag{34}$$

and thus $X \to \frac{m_1}{|e_1|} e_2^2 \to +\infty$.

(C) One charge only ($e_1 = 0$)

This case is **always stable**. Indeed, considering $e_1 = 0$, the stability condition (32) becomes:

$$l_0 + m_2 > m_1. \tag{35}$$

Then, considering the separability condition, which is now:

$$l_0 > \sigma_1 = m_1, \tag{36}$$

it is immediate to see that (35) is always true.

Remarks

- The most of the cases are stable
- The only unstable case we found is the small naked RN near the RN black hole
- The one-charge case is always stable
- If we consider the configurations used in the electric force lines plot, we find that they are all stable, except the case with a small naked charge near the RN black hole. That agrees with Bonnor limit.

Essential References

M. Pizzi, A. Paolino, *Electric force lines of the double Reissner-Nordstrom exact solution.*, IJMP, in press; arXiv:0804.0541

M. Pizzi, A. Paolino, *Stability in the Alekseev-Belinski solution.*, to be published.

G.A. Alekseev, V.A. Belinski. *Equilibrium configuration of two charged masses in general relativity.*, Phys. Rev. D, July 25 2007; arXiv:0706.1981

G.A. Alekseev, V.A. Belinski. *Proc. of XI Marcel Grossman Meeting (Berlin, July 2006), arXiv:gr-qc/0710.2515, 2007.*,

Other related references:

J.A. Wheeler, *Geometrodynamics*, edited by C. and B. De Witt, 1973

R.S. Hanni, R. Ruffini. *Phys. Rew. D, 8 (10), pagg 3259-3265*, 1973.

B. Linet *J. Phys. A, Vol 9, No. 7, 1976*

G. Alekseev *Sov. Phys. Dokl.,30,565 (1985)*

W. B. Bonnor. *Class. Quantum Grav. 10, 2077-2082*, (1993)

D. Bini, A. Geralico, R. Ruffini. *Phys. Rew. D, 75, 044012 (2007).*

D. Bini, A. Geralico, R. Ruffini. *Phys. Lett. A 360, 515 (2007).*

D. Bini, A. Geralico, R. Ruffini. *Phys. Rev. D 77, 064020 (2008).*