From the Einstein-Cartan to the Ashtekar-Barbero formulation of gravity and a possible interpretation of the Immirzi parameter

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- Motivations: The Immirzi parameter is a free constant that appears in the spectra of area and volume of LQG. Many attempts have been made to understand the origin of this ambiguity[a], here we face the problem from a topological point of view, focusing the attention on the role played by the Fermions fields in this context.

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- Goal: The Immirzi parameter is a quantization ambiguity connected with the topological structure of the quantum configuration space in analogy with the θ-angle of QCD.

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- Goal: The Immirzi parameter is a quantization ambiguity connected with the topological structure of the quantum configuration space in analogy with the θ-angle of QCD.
- On the way: I will be demonstrating that the presence of Fermions allows us to introduce a topological density called Nieh-Yan term, which links the E-C theory to the A-B one. It plays a role analogous to that played by the Chern-Simons functionals in Yang-Mills gauge theories and its presence is connected with the particular structure the LGT group has in temporal gauge fixed gravity.

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- Einstein-Cartan theory: The minimal coupling between gravity and spinor matter fields is described, then the canonical theory is constructed and the temporal gauge fixing discussed.
- Large gauge transformations: The system is canonically quantized and the structure of the effective quantum configuration space is studied using a particular Chern-Simons functional containing torsion. The role of the de Sitter group is clarified specifying its role as generator of a particular class of large gauge transformations.

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- Non-minimal action and the classical role of the Immirzi parameter: some motivations to introduce a non-minimal action are provided, the role of the Nieh-Yan class clarified and the Immirzi parameter is compared with the θ-angle of QCD.

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- Ashtekar-Barbero constraints: Rescaling the wave functional of the E-C theory by the Nieh-Yan functional, a modification in the quantum operators is generated, we will be demonstrating that this modification naturally leads to the Ashtekar-Barbero constraints with matter.

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- Non-minimal action and the classical role of the Immirzi parameter: some motivations to introduce a non-minimal action are provided, the role of the Nieh-Yan class clarified and the Immirzi parameter is compared with the θ-angle of QCD.
- Ashtekar-Barbero constraints: Rescaling the wave functional of the E-C theory by the Nieh-Yan functional, a modification in the quantum operators is generated, we will be demonstrating that this modification naturally leads to the Ashtekar-Barbero constraints with matter.
- Concluding remarks: The obtained results are briefly summarized and a comment on the appearance of the Immirzi parameter in the spectra of non-perturbative operators of a fully Lorentz covariant theory will conclude my presentation.

General Remarks (Torsion)

The II Cartan structure equation is

$$de^a + \omega^a{}_b \wedge e^b = T^a \,,$$

the solution is $\omega^a{}_b = \overset{\circ}{\omega}{}^a{}_b\,(e) + K^a{}_b$, where $T^a = K^a{}_b \wedge e^b$.

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Torsion can be decomposed according to the proper Lorentz group by introducing the following irreducible components:

1. the trace vector $T = e^b \lrcorner T_b$;
2. the pseudo-trace axial vector $S = 3!\star\left(e^b \wedge T_b\right)$
3. and the 2-form q^a, satisfying: $e_a \lrcorner q^a = 0$ and $e_b \wedge q^b = 0$.

We have:

$$\boxed{T^a = \frac{1}{3}\, e^a \wedge T - \frac{1}{3}\star(e^a \wedge S) + q^a}$$

Finally we note that $R^{ab} = \overset{\circ}{R}{}^{ab} + d^{(\overset{\circ}{\omega})} K^{ab} + K^a_{\ c} \wedge K^{cb}$.

General Remarks (Nieh-Yan 4-form)

The II Cartan structure equation implies the cyclic Bianchi identity

$$de^a + \omega^a{}_b \wedge e^b = d^{(\omega)} e^a = T^a \quad \Longrightarrow \quad R^a{}_b \wedge e^b = d^{(\omega)} T^a \,. \tag{1}$$

Considering the Nieh-Yan 4-form

$$N(e,\omega) = T^a \wedge T_a - R^{ab} \wedge e_a \wedge e_b \,, \tag{2}$$

we have:

$$N(e,\omega) = d^{(\omega)} e^a \wedge T_a - e_a \wedge d^{(\omega)} T^a = d^{(\omega)} \left(e_a \wedge T^a\right) = d \left(e_a \wedge T^a\right) . \tag{3}$$

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The Nieh-Yan 4-form is the only exact 4-form containing torsion.

Over a compact manifold, the integral of the Nieh-Yan form is:

$$\int \left(T^a \wedge T_a - e_a \wedge e_b \wedge R^{ab} \right) = \frac{L^2}{2} \left[P_4 \left(SO(5) \right) - P_4 \left(SO(4) \right) \right] , \tag{8}$$

where $P_4 = \int_{M^4} R^{ab} \wedge R_{ab}$ denotes the 4-dim Pontryagin classes.

Einstein-Cartan theory

We can describe a system of spin-$1/2$ fields coupled to gravity via the Einstein-Cartan action:

$$S_{EC}\left(e,\omega,\psi,\overline{\psi}\right)=\frac{1}{2}\int e_a\wedge e_b\wedge\star R^{ab}+\frac{i}{2}\int\star\, e_a\wedge\left(\overline{\psi}\gamma^a\mathcal{D}\psi-\overline{\mathcal{D}\psi}\gamma^a\psi\right),$$

where the covariant derivatives are defined as:

$$\mathcal{D}\psi=d\psi-\frac{i}{4}\,\omega^{ab}\Sigma_{ab}\psi \qquad \text{and} \qquad \overline{\mathcal{D}\psi}=d\overline{\psi}+\frac{i}{4}\,\overline{\psi}\Sigma_{ab}\omega^{ab}.$$

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Spinor fields generate torsion, the variation with respect to ω^{ab} gives

$$d^{(\omega)}e^a=T^a=\star\left(e^a\wedge e_b J^b_{(A)}\right) \quad \text{where} \quad J^d_{(A)}=\overline{\psi}\gamma^d\gamma^5\psi\,.$$

The unique solution is:

$$\omega^{ab}\left(e,\psi,\overline{\psi}\right)=\overset{\circ}{\omega}{}^{ab}\left(e\right)+\frac{1}{4}\,\epsilon^{ab}{}_{cd}e^c J^d_{(A)}.$$

E-C theory (effective action)

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We can pull back the E-C action on the solution of the structure equation:

$$
\begin{aligned}
S_{J-J}\left(e,\psi,\overline{\psi}\right) = & \frac{1}{4}\int \epsilon_{abcd}\, e^a \wedge e^b \wedge \mathring{R}^{\,cd} \\
& + \frac{i}{2}\int \star\, e_a \wedge \left(\overline{\psi}\gamma^a \mathring{\mathcal{D}}\psi - \overline{\mathring{\mathcal{D}}\psi}\gamma^a\psi\right) \\
& + \frac{3}{16}\int dV\, \eta_{ab} J^a_{(A)} J^b_{(A)}\,,
\end{aligned}
$$

well known as Einstein-Cartan effective action

digression...

Let us consider the Holst action:

$$S(e,\omega) = \frac{1}{2}\int e_a \wedge e_b \wedge \left(\star R^{ab} - \frac{1}{\beta} R^{ab}\right)\,, \tag{9}$$

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In pure gravity, the dynamics of the gravitational field is classically not affected by the Holst modification.

In the presence of minimally coupled fermions the situation changes:

The Bianchi cyclic identity becomes:

$$R^a{}_b \wedge e^b = d^{(\omega)} T^a , \quad (12)$$

as a consequence the Holst modification no longer vanishes and the classical theory results to be modified.

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- The minimal coupling of fermions to the gravitational field described by the Holst action generates a modification with respect to the Einstein-Cartan action;
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- it does not work for $\beta = \pm i$.

Some comments are in order:

- The minimal coupling of fermions to the gravitational field described by the Holst action generates a modification with respect to the Einstein-Cartan action;
- the Immirzi parameter acquires a classical meaning;
- the minimal approach lacks a clear geometrical explanation;
- it does not work for $\beta = \pm i$.

The above points would encourage to search for a different and geometrically well motivated action describing the interaction between spin-1/2 fields and gravity in analogy with the Holst approach.

Non-minimal action (construction)

It is worth noting that the Holst action can be rewritten as:

$$S(e,\omega) = \frac{1}{16}\int tr\left(1 - \frac{i}{\beta}\gamma^5\right) e \wedge e \wedge \star R \qquad (13)$$

where $e \wedge e = e^a \wedge e^b \Sigma_{ab}$ and $R = R^{ab}\Sigma_{ab}$.

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So we can imagine to introduce the same modification in the Dirac action, which becomes

$$S(\psi,\overline{\psi}) = \frac{i}{2}\int \star e_a \wedge \left(\overline{\psi}\gamma^a \mathcal{S}^5_{(\beta)} D\psi + h.c.\right)\,, \tag{16}$$

more explicitly the result is the following gravity-matter action:

Non-minimal action

$$S\left(e,\omega,\psi,\overline{\psi}\right) = \frac{1}{4}\int\left(\epsilon_{abcd}\, e^a\wedge e^b\wedge R^{cd} - \frac{2}{\beta}\, e_a\wedge e_b\wedge R^{ab}\right)$$
$$+\frac{i}{2}\int\star\, e_a\wedge\left[\overline{\psi}\gamma^a\left(1-\frac{i}{\beta}\gamma_5\right)\mathcal{D}\psi - \overline{\mathcal{D}\psi}\left(1-\frac{i}{\beta}\gamma_5\right)\gamma^a\psi\right],$$

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THIS ACTION IS DYNAMICALLY EQUIVALENT TO THAT OF THE EINSTEIN-CARTAN THEORY

Non-minimal action (features)

The non-minimal action[a] is characterized by the following features:

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Let me show you that the resulting effective theory is equivalent to E-C.

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Non-minimal action (effective theory)

We can calculate the irreducible components of torsion in the general case by variating the action with respect to the 1-form ω^{ab}, we have:

$$T = \frac{3}{4\alpha}\left(\frac{\alpha\beta-\beta^2}{\beta^2+1}\right)e_b J^b_{(A)}, \qquad S = -\frac{3\beta}{\alpha}\,\frac{\alpha\beta+1}{\beta^2+1}e_a J^a_{(A)}, \qquad q^c = 0.$$

They depend both on α and β. But...

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For $\alpha = \beta$ they reduce to

$$T = 0, \qquad S = -3\, e_a J^a_{(A)}, \qquad q^c = 0,$$

which correspond exactly to those coming out from the Einstein-Cartan theory. So, from now on we consider only the case $\alpha = \beta$.

Non-minimal action (geometrical aspects)

Consider: $S_{N-Y} = -\frac{1}{2\beta}\int \left[R^{ab}\wedge e_a\wedge e_b - \star\, e_a \wedge \left(\overline{\psi}\gamma^a\gamma^5\mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^5\gamma^a\psi\right)\right]$

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Using $\left[\gamma^a, \Sigma^{bc}\right] = 4\, i\eta^{a[b}\gamma^{c]}$, we can write:

$$\star\, e_a\wedge\left(\overline{\psi}\gamma_5\gamma^a\mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^a\gamma_5\psi\right) = d\left(\star\, e_a J^a_{(A)}\right) - \star\, e_a\wedge K^a{}_b J^b_{(A)}\,.$$

Considering that $T^a = \star\left(e^a\wedge e_b J^b_{(A)}\right) \implies T^a\wedge e_a = -\frac{1}{2}\star\, e_a J^a_{(A)}$ and

$$T^a\wedge T_a = K^a{}_b\wedge e^b\wedge\star\left(e_a\wedge e_c J^c_{(A)}\right) = \frac{1}{2}\epsilon_{acdf}K^a{}_b\wedge e^b\wedge e^d\wedge e^f J^c_{(A)} \qquad (18)$$

$$= -\frac{1}{2}\epsilon_{acdf}\epsilon^{gbdf}K^a{}_b\wedge\star e_g J^c_{(A)} = 2\delta^{[g}_a\delta^{b]}_c K^a{}_b\wedge\star e_g J^c_{(A)} = -\star\, e_a\wedge K^a{}_b J^b_{(A)}.$$

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Calculating, we finally obtain

$$S_{N-Y} = -\frac{1}{2\beta}\int\left[R^{ab}\wedge e_a\wedge e_b - T^a\wedge T_a + 2d\left(e_a\wedge T^a\right)\right] = -\frac{1}{2\beta}\int d\left(e_a\wedge T^a\right)$$

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REMARKS:

- the proposed action describes the same classical dynamics of the Einstein-Cartan action;
- the Immirzi parameter disappears from the effective classical action;
- it generalizes the Holst approach to the presence of fermions;
- the non-minimal action suggests a possible physical interpretation of the Immirzi parameter in analogy with the parameter θ of Yang-Mills gauge theories.

E-C theory (canonical formulation)

Consider again the Einstein-Cartan action

$$S_{EC}\left(e,\omega,\psi,\overline{\psi}\right) = \frac{1}{2}\int\limits_{M} e_a \wedge e_b \wedge \star R^{ab} + \frac{i}{2}\int\limits_{M} \star\, e_a \wedge \left(\overline{\psi}\gamma^a \mathcal{D}\psi - \overline{\mathcal{D}\psi}\gamma^a\psi\right),$$

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Assuming that space-time is a globally hyperbolic manifold

$$\Downarrow$$

Geroch theorem (1970)

On the space-time $(M, g_{\mu\nu})$ a global "time" function can be chosen such that each surface of constant t is a Cauchy surface. Thus M can be foliated by Cauchy surfaces and the topology of M is $\mathbb{R}\times\Sigma$, where Σ denotes any Cauchy surfaces.

3+1 splitting procedure...

Let us introduce the so called deformation vector t_μ, defined as

$$t^\mu \nabla_\mu t = 1$$

geometrically represents the “flow of time” throughout space-time, namely the directional derivative it generates corresponds to an increment in label time t.

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geometrically represents the “flow of time” throughout space-time, namely the directional derivative it generates corresponds to an increment in label time t. The deformation vector t^μ generates a one parameter group of diffeomorphisms

$$\phi_t : \mathbb{R} \times \sigma \rightarrow M \quad \text{as} \quad (t;x) \rightarrow y^\mu(t;x) := y^\mu_t(x), \quad \text{where} \quad \Sigma^3_t \stackrel{def}{=} y^\mu_t(x)\,,$$

A one-parameter family of embeddings is equivalent to a one-parameter group of diffeomorphisms

They are generally called “embedding diffeomorphisms”. The embedding diffeomorphisms is left completely arbitrary and is used for recasting the action into a 3+1-form.

Using the embedding diffeomorphisms we can rewrite the Einstein-Cartan action in a 3+1 form:

$$S_{EC} \xrightarrow{\phi_t} S_{3+1} = \int_{\mathbb{R}\times\sigma} dt d^3x \, \mathcal{L}_{3+1}$$

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We can now calculate the

CONJUGATED MOMENTA

$$\boxed{\pi^{\alpha}_{ab} = -e\, e^t_{[a} e^{\alpha}_{b]}\,, \quad \overline{\Pi}_A = \frac{i}{2}\, e\, e^t_a \overline{\psi}_B \left(\gamma^a\right)^B{}_A\,, \quad \Omega^B = -\frac{i}{2}\, e\, e^t_a \left(\gamma^a\right)^B{}_A \psi^A\,,}$$

respectively associated to the dynamical fields ω^{ab}_{α}, ψ^A and $\overline{\psi}_B$.

After having recognized the six primary constraints

$$C^{\alpha\beta} := \frac{1}{2}\,\epsilon^{abcd}\pi_{ab}^{(\alpha}\pi_{cd}^{\beta)} \approx 0\,,$$

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Specifically, after having equipped the phase space with the following

SYMPLECTIC STRUCTURE

$$\left\{\omega^{ab}_{\alpha}(t,x),\pi^{\beta}_{cd}(t,x')\right\} = \delta^{\beta}_{\alpha}\delta^{ab}_{cd}\delta^{(3)}(x-x')\,,$$

$$\{\psi^{A}(t,x),\overline{\Pi}_{B}(t,x')\}_{+} = \delta^{A}_{B}\delta^{(3)}(x-x')\,,$$

...we can extract the following full set of constraints

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FIRST CLASS CONSTRAINTS

$$\mathcal{G}_{ab} := D_\alpha \pi^\alpha_{ab} - \frac{i}{4}\,\overline{\Pi}\Sigma_{ab}\psi \approx 0\,,$$
$$\mathcal{C}_\alpha := \pi^\beta_{ab} R_{\alpha\beta}{}^{ab} - \overline{\Pi} D_\alpha \psi \approx 0\,,$$
$$\mathcal{C} := \frac{1}{2}\,\pi^\alpha_{ac}\pi^{\beta c}{}_b R_{\alpha\beta}{}^{ab} + i\pi^\alpha_{ab}\overline{\Pi}\Sigma^{ab} D_\alpha\psi \approx 0\,,$$

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SECOND CLASS CONSTRAINTS

$$C^{\alpha\beta} := \frac{1}{2}\,\epsilon^{abcd}\pi^{(\alpha}_{ab}\pi^{\beta)}_{cd} \approx 0\,,$$

$$D^{\alpha\beta} := \frac{1}{2}\epsilon^{abcd}\pi^\gamma_{ag}\pi^{(\alpha|g|}{}_b D_\gamma \pi^{\beta)}_{cd} - \frac{1}{4}\epsilon^{ab}{}_{cd}\pi^{(\alpha}_{fg}\pi^{\beta)}_{ab}\overline{\Pi}\Sigma^{fg}\Sigma^{cd}\psi \approx 0\,.$$

Temporal gauge fixing

In order to solve the second class constraints, we should replace the Poisson brackets with the Dirac ones, but this complicated procedure does not concern our scopes here and, as usual, we proceed to simplify the problem by partially fixing the gauge.

We fix $\pi^{\alpha}_{ij} = 0$, which corresponds to the so called temporal gauge; with this choice the gauge symmetry reduces from the full local Lorentz rotations to the subgroup of spatial rotations.

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We fix $\pi^{\alpha}_{ij} = 0$, which corresponds to the so called temporal gauge; with this choice the gauge symmetry reduces from the full local Lorentz rotations to the subgroup of spatial rotations.

As a consequence of the partial gauge fixing, we get the

REDUCED SYMPLECTIC FORM

$$\left\{K^i_\alpha(t,x), E^\beta_k(t,x')\right\} = \delta^\beta_\alpha \delta^i_k \delta^{(3)}(x-x')$$

$$\{\psi^A(t,x), \overline{\Pi}_B(t,x')\}_+ = \delta^A_B \delta^{(3)}(x-x') \,.$$

The main consequence of the partial gauge fixing is the reduction of the full set of constraints to the following seven first class constraints:

$$\mathcal{R}_i := \epsilon_{ij}{}^{k} K^j_\alpha E^\alpha_k - \frac{i}{4}\,\overline{\Pi}\Sigma_i\psi + \frac{i}{4}\,\overline{\psi}\Sigma_i\Omega \approx 0\,,$$

$$\mathcal{C}_\alpha := 2E^\beta_i \mathcal{D}_{[\alpha} K^i_{\beta]} - \overline{\Pi}\mathcal{D}_\alpha\psi - \mathcal{D}_\alpha\overline{\psi}\Omega \approx 0\,,$$

$$\mathcal{C} := \frac{1}{2}E^\alpha_i E^\beta_j \left(\epsilon^{ij}{}_k R^k_{\alpha\beta} + 2K^i_{[\alpha}K^k_{\beta]}\right) + iE^\alpha_i\left(\overline{\Pi}\Sigma^{0i}\mathcal{D}_\alpha\psi - \overline{\mathcal{D}_\alpha\psi}\Sigma^{0i}\Omega\right)$$
$$-\frac{i}{4}\epsilon^i{}_{kl}E^\alpha_i K^k_\alpha\left(\overline{\Pi}\Sigma^l\psi - \overline{\psi}\Sigma^l\Omega\right) - \frac{1}{4}E^\alpha_k K^k_\alpha\left(\overline{\Pi}\psi + \overline{\psi}\Omega\right) \approx 0\,,$$

notations: $E^\alpha_i = \pi^\alpha_{0i}\,, \quad K^j_\beta = \omega^{0j}_\beta\,, \quad \Sigma_i = \frac{1}{2}\epsilon_i{}^{jk}\Sigma_{jk}\,, \quad R^i = \frac{1}{2}\epsilon^i{}_{jk}\,{}^{(3)}R^{jk}$.

Moreover, by solving the second class constraints, we get the

compatibility condition

$$D_\alpha E^\beta_i = 0$$

Quantization procedure

The temporal gauge fixing allows us to reduce the full set of first and second class constraints to a set of first class constraints only, so that we can quantize the system by adopting the Dirac procedure, i.e. the constraints are directly implemented in the quantum theory by requiring that the state functional be annihilated by their operator representation.

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The wave function depends on half of the elementary variables. Once chosen a suitable quantum configuration space, namely the polarization, we have to equip it with the structure of a Hilbert space $\mathcal{H}(L_2, d\mu)$ called auxiliary Hilbert space.

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We have to require that the operator representation of the canonical variables, linearly acting on the auxiliary Hilbert space, generates an irreducible representation of the canonical commutation relation, i.e.

$$\left[\widehat{Q}(h), \widehat{P}(f)\right] = i\hbar \{\widehat{Q(h), P(f)}\}.$$

Quantization procedure (polarization)

Let me assume as coordinates on the quantum configuration space the following fields: E_i^α and ψ^A, so that the wave function is

$$\Phi = \Phi\left(E, \psi\right) \ .$$

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$$\widehat{\mathcal{R}}_i \Phi\left(E, \psi\right) = 0\,, \qquad \widehat{\mathcal{C}}_\alpha \Phi\left(E, \psi\right) = 0\,, \qquad \widehat{\mathcal{C}}\, \Phi\left(E, \psi\right) = 0\,,$$

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$$\Downarrow$$

Invariance under local internal symmetry

Whereas an internal symmetry is present, the state functional is invariant under small gauge transformations, i.e. the automorphisms of the quantum configuration space in the connected component of the identity.

Large gauge transformations

So the request that the state functional be annihilated by the quantum rotations operator $\widehat{\mathcal{R}}_i$ corresponds to the invariance of the state functional itself under small internal gauge transformations, but no hint is provided by the canonical theory about its behavior under the Large gauge transformations.

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$$\boxed{w(g) = n \neq 0}$$

In order to define the winding number and to review some useful concepts, let me refer to the specific case of a gauge theory with an internal $SU(N)$ symmetry...

LGT (winding number)

Consider the 3-sphere S^3 obtained by the one point compactification of $\mathbb{R}^3$, every element $g(x) \in SU(N)$ of the internal gauge group represents a continuous map

$$g(x) : S^3 \longrightarrow SU(N),$$

characterized by a number of disconnected components, specifically we have:

$$\Pi_3 \left(SU(N) \right) \simeq \mathbb{Z}.$$

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$$\Pi_3 \left(SU(N) \right) \simeq \mathbb{Z}.$$

We can compute explicitly the winding number by calculating the Cartan-Maurer integral

$$w(g) = \frac{1}{24\pi} \int \left(g^{-1} dg \right) \wedge \left(g^{-1} dg \right) \wedge \left(g^{-1} dg \right) \ ,$$

which tells us how many times $g(x)$ winds around the non-contractible 3-sphere S^3 in the $SU(N)$ manifold as x ranges over the 3-sphere S^3.

LGT (digression)

Consider an $SU(N)$ Yang-Mills gauge theory in its canonical form. The dynamics is described by the Hamiltonian,

$$H = \int d^3x\, tr \left[N \left(\frac{1}{2\sqrt{h}} \pi^\alpha \pi_\alpha + \frac{\sqrt{h}}{4} F_{\alpha\beta} F^{\alpha\beta} \right) - N^\beta \pi^\alpha F_{\alpha\beta} \right] \, ,$$

$\pi_\alpha = \frac{\sqrt{h}}{N} (\partial_t A_\alpha - \partial_\alpha A_t) + \sqrt{h} \frac{N^\beta}{N} F_{\alpha\beta}$ is the conjugate momentum to A.

The accessible part of the phase space is determined by the Gauss law

$$G_i := D_\alpha \pi_i^\alpha = \partial_\alpha \pi_i^\alpha + f_{ij}{}^k A_\alpha^j \pi_k^\alpha \approx 0 \, .$$

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$$G_i := D_\alpha \pi^\alpha_i = \partial_\alpha \pi^\alpha_i + f_{ij}{}^k A^j_\alpha \pi^\alpha_k \approx 0 \, .$$

Let me suppose to quantize the theory adopting the Dirac procedure and let $\phi = \phi(A)$ be the state functional describing the quantum system, we require that:

$$\boxed{\widehat{G}_i \phi(A) = 0}$$

LGT (behavior of the state functional)

As previously remarked, the previous equation obliges the state functional to be invariant under small gauge transformations. In this respect, we assume that $\hat{\mathcal{G}}_n$ represents the large gauge transformations operator.

Since the Hamiltonian is (fully) gauge invariant, one can construct a base of eigenstates diagonalizing simultaneously the Hamiltonian and the operator $\hat{\mathcal{G}}_n$.

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In other words, the following eigenvalues equation

$$\hat{\mathcal{G}}_n \phi(A) = \phi(A^{(g_n)}) = \exp\{i\theta n\}\,\phi(A)\,,$$

where n is the numerical value of the winding number, represents a super-selection rule for the eigenstates of the theory, each one marked by a different winding number. This is equivalent to have n different vacuum states $|n\rangle$

$$\hat{\mathcal{G}}_1\,|n\rangle = |n+1\rangle\ . \tag{24}$$

Moreover a one parameter ambiguity is introduced in the quantum theory.

Chern-Simons functionals

It is well known that it is possible to obtain a fully gauge invariant state functional by rescaling it by the exponential of the Chern-Simons functionals.

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Consider the following functional of the $SU(N)$ group valued connection A

$$\mathcal{Y}(A) = \frac{1}{16\pi} \int tr \left(F \wedge A + \frac{2}{3} A \wedge A \wedge A \right) ,$$

it is possible to demonstrate that the following relation holds:

$$\mathcal{Y}\left(A^g\right) = \mathcal{Y}\left(A\right) + w(g) \, .$$

Invariant state functional

So, let us rescale the wave functional by the exponential of the Chern-Simons functional and verify that it is an eigenfunction of the large gauge transformations operator with eigenvalues independent of the winding number:

$$\phi'(A) = \exp\{-i\theta\mathcal{Y}(A)\}\,\phi(A)\,,$$

thus, we have

$$\begin{aligned}\mathcal{G}_n\phi'(A) &= \mathcal{G}_n\left[\exp\{-i\theta\mathcal{Y}(A)\}\,\phi(A)\right]\\ &= \exp\{-i\theta\left[\mathcal{Y}(A)+w(g)\right]\}\cdot\exp\{i\theta n\}\,\phi(A) = \phi'(A)\,.\end{aligned}$$

In other words, it is possible to choose a fully gauge invariant vacuum, namely $|\theta\rangle = \sum_n exp(-in\theta)\,|n\rangle$.

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In other words, it is possible to choose a fully gauge invariant vacuum, namely $|\theta\rangle = \sum_n exp(-in\theta)\,|n\rangle$.
Consequently we have:

$$\pi'_\alpha\phi'(A) = e^{-i\theta\mathcal{Y}(A)}\pi'_\alpha e^{i\theta\mathcal{Y}(A)}\phi'(A) = -i\left[\frac{\delta}{\delta A^\alpha} - \frac{i\theta}{8\pi^2}\epsilon_\alpha^{\ \beta\gamma}F_{\beta\gamma}\right]\phi'(A)\,.$$

Modified Hamiltonian

The immediate consequence of the modification in the expression of the momentum operator is that the Hamiltonian changes too, specifically we have:

$$H' = \int d^3x \, tr \left[N \left(\frac{1}{2\sqrt{h}} \left(\pi_\alpha - \frac{\theta}{8\pi^2} \epsilon_\alpha{}^{\beta\gamma} F_{\beta\gamma} \right)^2 + \frac{\sqrt{h}}{4} F_{\alpha\beta} F^{\alpha\beta} \right) - N^\beta \pi^\alpha F_{\alpha\beta} \right] ,$$

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the presence of a pseudo-vectorial object implies that the Hamiltonian H' fails to be P and CP invariant.
We could have obtained the same result if we initially started from the following action:

$$S(A, \partial A) = \frac{1}{16\pi}\int \star F \wedge F + \frac{\theta}{16\pi}\int F \wedge F\,,$$

namely adding a θ dependent topological term to the usual Yang-Mills action.

...back to gravity...

The same concepts we introduced in Yang-Mills theories can be used in gravity. We recall that the rotational constraint,

$$\boxed{\widehat{R}_i \Phi(E, \psi) = 0}$$

plays the same role of the Gauss constraints of YM gauge theories.

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Let me indicate with $\widehat{\mathcal{G}}$ the large gauge operator... but, above all,

Question: are the large spatial rotations the only large gauge transformations involved in this problem?

...back to gravity...

My answer is: NO!

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Remark: The temporal gauge reduces the gauge group from $SO(3,1)$ to $SO(3)$, fixing the small component of $SO(3,1)/SO(3)$.

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This gauge fixing is connected with the appearance of the Immirzi parameter in the quantum theory

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The Immirzi parameter, in fact, as previously demonstrated, appears in front of the Nieh-Yan term. This behavior is analogous to that of θ previously described.

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This gauge fixing is connected with the appearance of the Immirzi parameter in the quantum theory

The Immirzi parameter, in fact, as previously demonstrated, appears in front of the Nieh-Yan term. This behavior is analogous to that of θ previously described.

It is possible to show that it exists a class of large gauge transformations which motivates the presence of the Nieh-Yan term in the classical theory.

back to gravity (consequences)

1. we can interpret the presence of the Immirzi parameter in the quantum theory;
2. we can construct a link between the E-C and A-B quantum theories, in view, also, of a possible classical limit of quantum gravity.

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The Nieh-Yan term is generated by a particular class of LGT. We have to consider the large component of the gauge fixed sector of the full Lorentz group. But, how can we do this in the present framework?

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The Nieh-Yan term is generated by a particular class of LGT. We have to consider the large component of the gauge fixed sector of the full Lorentz group. But, how can we do this in the present framework?

By introducing the

MacDowell-Mansouri connection

$$\Gamma^{ab} = \begin{pmatrix} \omega^{ij} & \ell_{Pl}^{-1}\, e^{i} \\ -\ell_{Pl}^{-1}\, e^{j} & 0 \end{pmatrix}.$$

...back to gravity...

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Consider a 2-dim manifold, on the tangent bundle acts the $SO(2)$ group in the natural way. Now imagine to construct a new bundle via the one point compactification of $\mathbb{R}^2$, namely the "bundle of tangent spheres": the group $SO(3)$ has a natural action on it. So, extending the gauge group means considering the parallel transport of spheres instead of the parallel transport of planes!

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What is the link with the MacDowell-Mansouri connection? The $SO(3)$ connection constructed above rotates 3-dim spheres, the small component of the rotation can be factorized in a $SO(2)$ rotation that leaves the tangent point P fixed and a translation of the tangent point itself. In other words we have:

$$\boxed{so(3) \simeq so(2) \oplus S^1}$$

Back to gravity (LGT)

Using the MacDowell-Mansouri connection we can construct the associated Chern-Simons functional

$$\mathcal{Y}(\Gamma) = \frac{1}{4}\int \left(F^{ab} \wedge \Gamma_{ab} + \frac{1}{3}\Gamma^{a}{}_{b} \wedge \Gamma^{b}{}_{c} \wedge \Gamma^{c}{}_{a} \right),$$

and we can note that...

we have that...

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and we can note that...

...factorizing the spatial rotations, namely

$$\mathcal{Y}(\Gamma) - \mathcal{Y}(\omega) = \frac{1}{4}\int \Big[F^{ab} \wedge \Gamma_{ab} + \frac{1}{3}\Gamma^a{}_b \wedge \Gamma^b{}_c \wedge \Gamma^c{}_a - \left(R^{ij} \wedge \omega_{ij} + \frac{1}{3}\omega^i{}_j \wedge \omega^j{}_k \wedge \omega^k{}_i \right) \Big]$$

we have that...

Nieh-Yan functional

The 3-form $\mathcal{Y}_{NY}(e,\omega) = \mathcal{Y}(\Gamma) - \mathcal{Y}(\omega)$ defined in the previous slide can be rewritten as

$$\mathcal{Y}_{NY} = \frac{1}{2}\int e_i \wedge T^i \,,$$

which is called Nieh-Yan functional and contains torsion.

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which is called Nieh-Yan functional and contains torsion.

In other words, the Nieh-Yan functional turns out to be the difference between two Chern-Simons functionals.

Rescaled functional

This suggests that rescaling the wave functional by the exponential of the Nieh-Yan functional corresponds to take into account the fact that the gauge fixing acts only on the small component of the boost sector of the initial gauge group, so that if we want to construct a state functional invariant under the full boost sector we have to consider

$$\Phi'(E,\psi) = \exp\left\{-\frac{i}{\beta}\,Y(E)\right\}\Phi(E,\psi)\,,$$

but this inevitably introduces a 1-parameter ambiguity in the theory.

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but this inevitably introduces a 1-parameter ambiguity in the theory. This rescaling modifies the momentum operators $\widehat{K}^i_\alpha$, $\overline{\Pi}_B$

$$\widehat{K}'^{\,i}_{\alpha}\Phi'(E,\psi) = \left(\widehat{K}^i_\alpha - \frac{1}{\beta e}\,\epsilon^{ij}{}_k E^\beta_j T^k_{\alpha\beta}\right)\Phi'(E,\psi)\,,$$

$$\widehat{\overline{\Pi}}'_B\Phi'(E,\psi) = \left(\widehat{\overline{\Pi}}_B - \frac{i}{\beta}\widehat{\overline{\Pi}}_A\left(\gamma^5\right)^A{}_B\right)\Phi'(E,\psi)\,,$$

generating the following modifications in the canonical constraints...

...we have

$$\mathcal{R}'_i = \frac{1}{\beta}\mathcal{D}_\alpha E^\alpha_i + \epsilon_{ij}{}^k K^j_\alpha E^\alpha_k - \frac{i}{4}\,\overline{\Pi} S^5_{(\beta)}\Sigma_i\psi \approx 0\,,$$

$$\mathcal{C}'_\alpha = 2E^\beta_i \mathcal{D}_{[\alpha}K^i_{\beta]} + \frac{1}{2\beta}\,\epsilon^{ij}{}_k E^\beta_i R_{\alpha\beta j}{}^k - \overline{\Pi} S^5_{(\beta)}\mathcal{D}_\alpha\psi \approx 0\,,$$

$$\mathcal{C}' = \frac{1}{2}E^\alpha_i E^\beta_k \left(\epsilon^{ik}{}_l R^l_{\alpha\beta} + 2K^i_{[\alpha}K^k_{\beta]}\right) + iE^\alpha_i \overline{\Pi} S^5_{(\beta)}\Sigma^{0i}\mathcal{D}_\alpha\psi$$
$$- \frac{i}{4}\epsilon^i{}_{kl}E^\alpha_i K^k_\alpha \overline{\Pi} S^5_{(\beta)}\Sigma^l\psi - \frac{1}{4}E^\alpha_k K^k_\alpha \overline{\Pi} S^5_{(\beta)}\psi \approx 0\,,$$

where $S^5_{(\beta)} = 1 - \frac{i}{\beta}\gamma^5$. It is worth noting that the gravitational contributions in above expressions are exactly those obtainable starting from the Holst action, while the matter contribution are those obtainable by the non-minimal coupling.

Provided that the new variables A^i_α are defined via the "symplecto-morphisms"

$$A^k_\alpha = \beta K^k_\alpha - \frac{1}{2}\epsilon_{ij}{}^k \omega^{ij}_\alpha \, ,$$

and, consequently, the new symplectic structure

$$\left\{ A^i_\alpha(t,x), E^\beta_k(t,x') \right\} = \beta \delta^\beta_\alpha \delta^i_k \delta^{(3)}(x - x') \, ,$$

is assumed...

...we exactly get <u>the Ahtekar-Barbero constraints with fermions</u>.

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