Recent Developments in Particles and Fields Motion within the Kaluza-Klein Picture

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SUMMARY

- 5D KALUZA-KLEIN MODEL IN VACUUM

- THE PROBLEM OF MATTER

- REVISED APPROACH TO MATTER

- KALUZA-KLEIN MODEL WITH MATTER

5D KALUZA KLEIN MODEL IN VACUUM

Starting hypotheses:

$\mathcal{M}^5 = \mathcal{M}^4 \otimes S^1$

Cylindricity: $\partial_5 J_{AB} = 0$

J_{55} is a scalar

Metrics:

$$J_{AB} = \begin{pmatrix} g_{\mu\nu} - \phi^2 (ek)^2 A_\mu A_\nu & -\phi^2 (ek) A_\mu \\ -\phi^2 (ek) A_\mu & -\phi^2 \end{pmatrix} \quad (ek)^2 = 4G\,;\ c = 1$$

Invariance with respect to:

$$\begin{cases} x^{5\prime} = x^5 + ek\Psi(x^\mu) \\ x^{\mu\prime} = x^{\mu\prime}(x^\nu) \end{cases}$$

5D covariance and 5D free falling universality (FFU) of particles are violated.

Fields Dynamics

Action:from the reduction of $S_{fields} = -\frac{1}{16\pi G_5}\int d^5x\sqrt{J}\,R^5$ we get

$$S_{fields} = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(\phi R - 2\Box\phi + \frac{1}{4}(ek)^2\phi^3 F_{\mu\nu}F^{\mu\nu})$$

where $G^{-1} = G_5^{-1}\int dx^5$.

Fields Equations:

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^\mu\partial^\nu\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi + 8\pi G\phi^2 T^{\mu\nu}_{em}$$

$$\nabla_\mu\left(\phi^3 F^{\mu\nu}\right) = 0$$

$$\Box\phi = -\frac{1}{4}\phi^3(ek)^2 F^{\mu\nu}F_{\mu\nu}$$

If we put $\phi = 1$ we recover Einstein-Maxwell dynamics. Such a condition has to be imposed *before* the variational procedure, otherwise we get the inconsistent result $F^{\mu\nu}F_{\mu\nu} = 0$.

THE PROBLEM OF MATTER

Ansatz

$$S_{particle} = -\hat{m}\int ds_5 \qquad ds_5^2 = ds^2 - \phi^2(ekA_\mu dx^\mu + dx^5)^2$$

it leads to : $P_A P^A = \hat{m}^2 \quad P^A = \hat{m} w^A$.

Motion equation:

$$\begin{cases} \frac{d}{ds} w_5 = 0 \\ \\ \frac{D}{Ds} u^\mu = F^{\mu\nu} u_\nu \left(\frac{ekw_5}{\sqrt{1+\frac{w_5^2}{\phi^2}}} \right) + \frac{1}{\phi^3}(u^\mu u^\nu - g^{\mu\nu})\partial_\nu \phi \left(\frac{w_5^2}{1+\frac{w_5^2}{\phi^2}} \right) \end{cases}$$

4D Dispersion relation:

$$P_A P^A = \hat{m}^2 \qquad \Rightarrow \qquad g^{\mu\nu}\Pi_\mu \Pi_\nu = \hat{m}^2 + \frac{P_5^2}{\phi^2}$$

where $\Pi_\mu = P_\mu - ekP_5 A_\mu$.

Klein Gordon

$$P_A P^A \zeta = \zeta \hat{m}^2 \qquad \zeta(x^\mu, x^5) = \eta(x^\mu) e^{iP_5 x^5}$$

$$\mathcal{L} = g^{\mu\nu}(-i\partial_\mu - P_5 ekA_\mu)\eta[(-i\partial_\nu - P_5 ekA_\nu)\eta]^+ - \left(\hat{m}^2 + \frac{P_5^2}{\phi^2}\right)\eta\eta^+$$

Properties of Couplings:

$$q = ekP_5 \qquad m^2 = \left(\hat{m}^2 + \frac{P_5^2}{\phi^2}\right)$$

$$\frac{q^2}{4Gm^2} = \frac{P_5^2}{\hat{m}^2 + \frac{P_5^2}{\phi^2}} = \frac{w_5^2}{1 + \frac{w_5^2}{\phi^2}}$$

Consider for simplicity $\phi = 1$:

$$\frac{q^2}{4Gm^2} = \frac{w_5^2}{1 + w_5^2} < 1 \qquad \frac{e^2}{4Gm_e^2} \sim 10^{42}$$

Taking into account the compactification:

$$P_{5(n)} = 2\pi n/L_5 \qquad m_{(n)}^2 = \hat{m}^2 + P_{5(n)}^2$$

Setting $P_{5min} = \frac{e}{\sqrt{4G}}$ we get $L_5 \sim 10^{-31}\, cm$ but values of mass are beyond the Planck scale.

This approach is not consistent with Lorenzian dynamics because it provides a bounded q/m and gives a huge massive modes spectrum.

REVISED APPROACH TO MATTER

$$^{5}\nabla_A T^{AB} = 0 \qquad \partial_5 T^{AB} = 0$$

After KK reduction we get a conserved current:

$$5) \quad \rightarrow \nabla_\mu \left(\phi T_5^\mu\right) = 0 \qquad \rightarrow j^\mu = ek\phi T_5^\mu$$

$$\mu) \quad \rightarrow \nabla_\rho (\phi T^{\mu\rho}) = -g^{\mu\rho} \left(\frac{\partial_\rho \phi}{\phi^2}\right) T_{55} + F^\mu_{\ \rho} j^\rho$$

For a point-like particle, after a Papapetrou expansion along a 4D world line X^μ,we have:

$$m\frac{Du^\mu}{Ds} = A(u^\rho u^\mu - g^{\mu\rho})\frac{\partial_\rho \phi}{\phi} + qF^{\mu\rho}u_\rho$$

$$\begin{aligned}
m &= \frac{1}{u^0}\int d^3x \sqrt{g}\,\phi T^{00} & \sqrt{g}\phi T^{\mu\nu} &= \int ds\, m\, \delta^4(x-X)u^\mu u^\nu \\
q &= ek\int d^3x \sqrt{g}\,\phi T_5^0 & ek\sqrt{g}\phi T_5^\mu &= \int ds\, q\, \delta^4(x-X)u^\mu \\
A &= u^0\int d^3x \sqrt{g}\,\frac{T_{55}}{\phi} & \frac{\sqrt{g}T_{55}}{\phi} &= \int ds\, A\, \delta^4(x-X)
\end{aligned}$$

The effective tensor for test particles is localized in 4D. The particle is not localized along the extra dimension.

Couplings:

m and q are not correlated via P_5; they are defined in terms of independent degrees of freedom (T^{00} and T_5^0) therefore we have no bound on q/m.

q: conserved - U(1) symmetry

A: not conserved - no symmetry requires it

m: not conserved

$$\frac{dm}{ds} = -\frac{A}{\phi}\frac{d\phi}{ds}$$

While in GR the single-pole equation of Papapetrou coincides with the geodesic equation, in the 5D KK model they are different. The reason has to be addressed to the cylindricity condition and to the violation of PE.

Canonical Formulation

$$S_{particle} = -\int m\, ds + q(A_\mu dx^\mu + \frac{dx^5}{ek}).$$

where $\partial_\mu m = -\frac{A}{\phi}\partial_\mu \phi$.

Dispersion Relation:

$$P_A P^A = m^2 - \frac{q^2}{(ek)^2\phi^2} \quad \Longleftrightarrow \begin{cases} \Pi_\mu \Pi^\mu = m^2 \\ P_5 = \frac{q}{ek} \\ \Pi_\mu = P_\mu - qA_\mu \end{cases}$$

The 4D disp.relation shows the proper mass term; the 5D disp.relation has now a counterterm which removes the tower of massive modes.

$$\mathcal{L} = g^{\mu\nu}(-i\partial_\mu - qA_\mu)\eta[(-i\partial_\nu - qA_\nu)\eta]^+ - m^2\eta\eta^+,$$

Charge is still quantized as consequence of compactification but mass is unaffected (no link with P_5).

KALUZA KLEIN THEORY WITH MATTER

We consider the full dynamics of 5D Einstein equations + 5D matter tensor :

we identify

$$T^{\mu\nu}_{matter} = T^{\mu\nu}\phi \qquad j^{\mu} = ek\phi T^{\mu}_{5}$$

and we get

$$G^{\mu\nu} = \frac{1}{\phi}\nabla^{\mu}\partial^{\nu}\phi - \frac{1}{\phi}g^{\mu\nu}\Box\phi + 8\pi G\phi^{2}T^{\mu\nu}_{em} + 8\pi G\frac{T^{\mu\nu}_{matt}}{\phi}$$

$$\nabla_{\nu}\left(\phi^{3}F^{\nu\mu}\right) = 4\pi j^{\mu}$$

$$\frac{1}{2}R + \frac{3}{8}\phi^{2}(ek)^{2}F^{\mu\nu}F_{\mu\nu} = 8\pi G\frac{T_{55}}{\phi^{2}}$$

arranging the last equation with the trace of the first:

$$\Box\phi = -\frac{1}{4}\phi^{3}(ek)^{2}F^{\mu\nu}F_{\mu\nu} + \frac{8}{3}\pi G\left(T_{matter} + 2\frac{T_{55}}{\phi}\right)$$

Note that the problem concerning $\phi = 1$ is now removed.

Simple scenarios (for $A_\mu = 0$)

$$\Box\phi = \frac{8}{3}\pi G\left(T_{matter} + 2\frac{T_{55}}{\phi}\right)$$

$$\frac{dm}{ds} = -\frac{A}{\phi}\frac{d\phi}{ds} \qquad m\frac{Du^\mu}{Ds} = A(u^\rho u^\mu - g^{\mu\rho})\frac{\partial_\rho\phi}{\phi}$$

$2T_{55} = -\phi T_{matter} \Longrightarrow \phi = 1$, $m = cost$, FFU $\rightarrow$ GR

$T_{55} = 0 \Longrightarrow m = cost$, FFU, ϕ variable

$A = \alpha m \Longrightarrow$ FFU, ϕ variable, m variable:

Noticeably, if we assume $A = \alpha m$ we recover FFU and we have

$$m = m_0\left(\frac{\phi}{\phi_0}\right)^{-\alpha}$$

CONCLUSIONS

- This approach offers a scenario to deal consistently with particles in the framework of the cylindrical and compactified KK model. The key point is the extension of the cylindricity hypothesis to the matter tensor, which becomes localized in 4D, i.e. the test particle is not localized in the extra dimension.

- Therefore it is possible to deal with the full theory, fields + matter source terms. Next steps: cosmological and spherical solution (In progress...).

- Some perspectives are related to the behavior of mass: a varying mass is interesting for various dark matter model.

[VL,G.Montani *Int.Journ.Mod.Phys.D* **4**, (2006) 559]

[VL,G.Montani *Dynamics of Matter in a 5D compactified KK Model* submitted to *CQG*]

[VL,G.Montani *Geometry and Matter Reduction in a 5D KK framework* to be submitted]

$$S_5 = -\frac{1}{16\pi {G_5}'} \int d^4x \sqrt{-g}\,(\phi R + 2\nabla_\mu \partial_\nu \phi + \frac{1}{4}(ek)^2 \phi^3 F_{\mu\nu} F^{\mu\nu})$$

$${G_5}'^{-1} = {G_5}^{-1} \int dx^5$$

$$G = \phi^{-1} {G_5}' \qquad 4G = (ek)^2 \phi^2 \Rightarrow ek^2 = \frac{4G_0}{\phi_0^2}$$

$$\frac{q}{m} = ek \frac{w_5}{\sqrt{1 + \frac{w_5^2}{\phi^2}}} \qquad \Rightarrow \frac{q_0}{\sqrt{4G_0} m_0} = \frac{\frac{w_5}{\phi_0}}{\sqrt{1 + \frac{w_5^2}{\phi_0^2}}} < 1$$

standard approach

In a synchronous frame, we have:

$$(H - ekP_5A_0)^2 = h^{ij}\Pi_i\Pi_j + \left(\hat{m}^2 + \frac{P_5^2}{\phi^2}\right)$$

where $\Pi_i = P_i - ekP_5A_i$ and we have $\Pi_i = \hat{m}w_i, P_5 = \hat{m}w_5$, being $w^A = \frac{dx^A}{ds_5}$. If we identify H with the time component of the momentum we can write a 4D covariant relation which holds in every frame:

$$g^{\mu\nu}\Pi_\mu\Pi_\nu = \hat{m}^2 + \frac{P_5^2}{\phi^2},$$

where $\Pi_0 = P_0 - ekP_5A_0$. Finally, we can rebuild a 5D dispersion relation: we define $P^A = \hat{m}w^A$. It yields:

$$\begin{aligned}
P^A &= (\Pi^\mu, P^5) \\
P_A &= (\Pi_\mu + P_5A_\mu, P_5) \\
\Pi^\mu &= g^{\mu\nu}\Pi_\nu \qquad P_5 = -\phi^2(A_\mu\Pi^\mu + P^5).
\end{aligned}$$

With such a definition we can reproduce the 4D relation by the following 5D dispersion relation :

$$P_AP^A = \hat{m}^2.$$

revised approach

In a synchronous frame the following formula holds:

$$(H - qA_0)^2 = h^{ij}\Pi_i\Pi_j + m^2,$$

where we have

$$\begin{aligned} P_5 &= q \\ \Pi_i &= P_i - qA_i = mu_i \qquad u^\mu = \frac{dx^\mu}{ds}. \end{aligned}$$

Identifying the Hamiltonian with the time component of the momentum we can rewrite the previous equation in a 4D manifestly covariant expression:

$$\Pi_\mu \Pi^\mu = m^2,$$

where $\Pi_0 = P_0 - qA_0$. Finally, we can rebuild a 5D dispersion relation: the calculus of P_μ, P_5 we got from the Lagrangian, uniquely defines a 5D vector P^A such that we have:

$$\begin{aligned} P^A &= (\Pi^\mu, P^5) \\ P_A &= (\Pi_\mu + P_5 A_\mu, P_5) \\ \Pi^\mu &= g^{\mu\nu}\Pi_\nu \qquad P_5 = -\phi^2(A_\mu\Pi^\mu + P^5). \end{aligned}$$

where $P_5 = q$, $\Pi^\mu = mu^\mu$. With such a definition, previous relations are provided by the following 5D dispersion relation :

$$P_A P^A = m^2 - \frac{q^2}{\phi^2}$$