On the “electric Meissner effect” in the field of a Reissner-Nordstrom black hole

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RN metric:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

$$f(r) = 1 - \frac{2\mathcal{M}}{r} + \frac{Q^2}{r^2}$$

horizons:

$$r_\pm = \mathcal{M} \pm \sqrt{\mathcal{M}^2 - Q^2} \equiv \mathcal{M} \pm \Gamma$$

e.m. field:

$$F = -\frac{Q}{r^2} dt \wedge dr$$

Perturbations on a RN spacetime

Einstein-Maxwell system of equations:

$$\tilde{G}_{\mu\nu} = 8\pi \left(T_{\mu\nu} + \tilde{T}^{\rm em}_{\mu\nu}\right)$$

$$\tilde{F}^{\mu\nu}{}_{;\nu} = 4\pi J^{\mu} \; , \qquad {}^{*}\tilde{F}^{\alpha\beta}{}_{;\beta} = 0$$

perturbed quantities:

$$\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$$

$$\tilde{F}_{\mu\nu} = F_{\mu\nu} + f_{\mu\nu}$$

$$\tilde{T}^{\rm em}_{\mu\nu} = \frac{1}{4\pi}\left[\tilde{g}^{\rho\sigma}\tilde{F}_{\rho\mu}\tilde{F}_{\sigma\nu} - \frac{1}{4}\tilde{g}_{\mu\nu}\tilde{F}_{\rho\sigma}\tilde{F}^{\rho\sigma}\right]$$

$$\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2}\tilde{g}_{\mu\nu}\tilde{R}$$

expansion up to the linear order:

$$\tilde{G}_{\mu\nu} \simeq G_{\mu\nu} + \delta G_{\mu\nu}$$

$$\tilde{T}^{\rm em}_{\mu\nu} \simeq T^{\rm em}_{\mu\nu} + \delta T^{\rm em}_{\mu\nu}$$

where

$$\tilde{F}^{\mu\nu}{}_{;\nu} \simeq f^{\mu\nu}{}_{;\nu} - \delta J^{\mu}_{\rm grav}$$

$$\begin{aligned}\delta G_{\mu\nu} &= -\frac{1}{2} h_{\mu\nu;\alpha}{}^{;\alpha} + k_{(\mu;\nu)} - R_{\alpha\mu\beta\nu} h^{\alpha\beta} - \frac{1}{2} h_{;\mu;\nu} + R^{\alpha}{}_{(\mu} h_{\nu)\alpha} \\ &\quad -\frac{1}{2} g_{\mu\nu} \left[k_{\lambda}{}^{;\lambda} - h_{;\lambda}{}^{;\lambda} - h_{\alpha\beta} R^{\alpha\beta} \right] - \frac{1}{2} h_{\mu\nu} R\end{aligned}$$

$$\delta T^{\rm em}_{\mu\nu} \equiv \delta T^{(h)}_{\mu\nu} + \delta T^{(f)}_{\mu\nu} \qquad \text{(e.m. induced gravitational perturbation)}$$

$$\delta T^{(h)}_{\mu\nu} = -\frac{1}{4\pi} \left[\left(F^{\alpha}{}_{\mu} F^{\beta}{}_{\nu} - \frac{1}{2} g_{\mu\nu} F^{\alpha\lambda} F^{\beta}{}_{\lambda} \right) h_{\alpha\beta} + \frac{1}{4} F^{\rho\sigma} F_{\rho\sigma} h_{\mu\nu} \right]$$

$$\delta T^{(f)}_{\mu\nu} = -\frac{1}{4\pi} \left[2 F^{\rho}{}_{(\mu} f_{\nu)\rho} + \frac{1}{2} g_{\mu\nu} F^{\rho\sigma} f_{\rho\sigma} \right]$$

$$\delta J^{\mu}_{\rm grav} = F^{\mu\rho}{}_{;\sigma} h_{\rho}{}^{\sigma} + F^{\rho\sigma} h^{\mu}{}_{\rho;\sigma} + F^{\mu\rho} \left(k_{\rho} - \frac{1}{2} h_{;\rho} \right) \quad \text{(grav. ind. e.m. pert.)}$$

$$k_{\mu} = h_{\mu\alpha}{}^{;\alpha} , \qquad h = h_{\alpha}{}^{\alpha}$$

Static perturbations by a point particle at rest

a point charge of mass m and charge q moving along a worldline $z^\alpha(\tau)$ with 4-velocity $U^\alpha = dz^\alpha/d\tau$ is described by:

$$T^{\mu\nu}_{\text{part}} = \frac{m}{\sqrt{-g}} \int \delta^{(4)}(x - z(\tau)) U^\mu U^\nu \, d\tau$$

$$J^{\mu}_{\text{part}} = \frac{q}{\sqrt{-g}} \int \delta^{(4)}(x - z(\tau)) U^\mu \, d\tau$$

charged particle at rest at the point $r = b$ on the polar axis $\theta = 0$ (and 4-velocity $U = f(r)^{-1/2}\partial_t$):

$$T^{\text{part}}_{00} = \frac{m}{2\pi b^2} f(b)^{3/2} \delta\left(r - b\right) \delta\left(\cos\theta - 1\right)$$

$$J^{0}_{\text{part}} = \frac{q}{2\pi b^2} \delta\left(r - b\right) \delta\left(\cos\theta - 1\right)$$

Tensor harmonic expansion of the fields

perturbed gravitational field (electric multipoles):

$$||h_{\mu\nu}|| = \begin{bmatrix} e^{\nu} H_0 Y_{l0} & H_1 Y_{l0} & h_0 \dfrac{\partial Y_{l0}}{\partial \theta} & 0 \\ \text{sym} & e^{-\nu} H_2 Y_{l0} & h_1 \dfrac{\partial Y_{l0}}{\partial \theta} & 0 \\ \text{sym} & \text{sym} & r^2 \left(K Y_{l0} + G \dfrac{\partial^2 Y_{l0}}{\partial \theta^2} \right) & 0 \\ \text{sym} & \text{sym} & \text{sym} & r^2 \sin^2 \theta \left(K Y_{l0} + G \cot \theta \dfrac{\partial Y_{l0}}{\partial \theta} \right) \end{bmatrix}$$

Regge-Wheeler gauge:

$$h_0 \equiv h_1 \equiv G \equiv 0$$

→

$$||h_{\mu\nu}|| = \begin{bmatrix} e^{\nu} H_0 Y_{l0} & H_1 Y_{l0} & 0 & 0 \\ H_1 Y_{l0} & e^{-\nu} H_2 Y_{l0} & 0 & 0 \\ 0 & 0 & r^2 K Y_{l0} & 0 \\ 0 & 0 & 0 & r^2 \sin^2\theta K Y_{l0} \end{bmatrix}$$

e.m. field:

$$||f_{\mu\nu}|| = \begin{bmatrix} 0 & \tilde{f}_{01}Y_{l0} & \tilde{f}_{02}\dfrac{\partial Y_{l0}}{\partial\theta} & 0 \\ \text{antisym} & 0 & \tilde{f}_{12}\dfrac{\partial Y_{l0}}{\partial\theta} & 0 \\ \text{antisym} & \text{antisym} & 0 & 0 \\ \text{antisym} & \text{antisym} & \text{antisym} & 0 \end{bmatrix}$$

source terms:

$$\sum_l A_{00}Y_{l0} = 16\pi T_{00}^{\text{part}} \, , \qquad \sum_l vY_{l0} = j^{0}_{\text{part}}$$

with

$$A_{00} \; = \; 8\sqrt{\pi}\frac{m\sqrt{2l+1}}{b^2}f(b)^{3/2}\delta\left(r-b\right)$$

$$v \; = \; \frac{1}{2\sqrt{\pi}}\frac{q\sqrt{2l+1}}{b^2}\delta\left(r-b\right)$$

first order perturbation equations:

$$\tilde{G}_{00} = -\frac{1}{2}\Big\{ e^{2\nu}\left[2K'' - \frac{2}{r}H_2{}' + \left(\nu' + \frac{6}{r}\right)K' - 2\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)(H_0 + H_2)\right]$$
$$-\frac{2e^{\nu}}{r^2}[(\lambda+1)H_2 - H_0 + \lambda K]\Big\}Y_{l0}$$

$$\tilde{G}_{11} = -\frac{1}{2}\Big\{\frac{2}{r}H_0{}' - \left(\nu' + \frac{2}{r}\right)K' + \frac{2e^{-\nu}}{r^2}[H_2 - (\lambda+1)H_0 + \lambda K]\Big\}Y_{l0}$$

$$\tilde{G}_{22} = \frac{r^2}{2}e^{\nu}\Big\{K'' + \left(\nu' + \frac{2}{r}\right)K' - H_0{}'' - \left(\frac{\nu'}{2} + \frac{1}{r}\right)H_2{}' - \left(\frac{3\nu'}{2} + \frac{1}{r}\right)H_0{}'$$
$$+2(\lambda+1)\frac{e^{-\nu}}{r^2}(H_0 - H_2) + \left(\nu'' + \nu'^2 + \frac{2\nu'}{r}\right)(K - H_2)\Big\}Y_{l0}$$
$$+\frac{1}{2}\Big\{H_0 - H_2\Big\}\frac{\partial^2 Y_{l0}}{\partial\theta^2}$$

$$\tilde{G}_{12} = -\frac{1}{2}\Big\{-H_0{}' + K' - \left(\frac{\nu'}{2} + \frac{1}{r}\right)H_2 - \left(\frac{\nu'}{2} - \frac{1}{r}\right)H_0\Big\}\frac{\partial Y_{l0}}{\partial\theta}$$

$$\tilde{G}_{01} = \Big\{\left[\frac{\lambda}{r^2} + \frac{e^{\nu}}{r}\left(\nu' + \frac{1}{r}\right)\right]H_1\Big\}Y_{l0}$$

$$\tilde{G}_{02} = \frac{e^{\nu}}{2}\{H_1{}' + \nu' H_1\}\frac{\partial Y_{l0}}{\partial\theta}$$

$$\lambda = \frac{1}{2}(l-1)(l+2)\ , \qquad e^{\nu} = f(r)$$

$$
\begin{aligned}
\tilde{T}_{00} &= -\frac{1}{8\pi}\left\{\frac{Q^2 e^{\nu} H_2}{r^4} + 2\frac{Q e^{\nu}\tilde{f}_{01}}{r^2}\right\} Y_{l0} \ , \\
\tilde{T}_{11} &= -\frac{1}{8\pi}\left\{\frac{Q^2 e^{-\nu} H_0}{r^4} - 2\frac{Q e^{-\nu}\tilde{f}_{01}}{r^2}\right\} Y_{l0} \ , \\
\tilde{T}_{22} &= \frac{r^2 e^{\nu}}{8\pi}\left\{\frac{Q^2 e^{-\nu} K}{r^4} - \frac{2Q e^{-\nu}}{r^2}\tilde{f}_{01}\right\} Y_{l0} \ , \\
\tilde{T}_{12} &= \frac{1}{8\pi}\left\{2\frac{Q e^{-\nu}\tilde{f}_{02}}{r^2}\right\}\frac{\partial Y_{l0}}{\partial\theta} \ , \\
\tilde{T}_{01} &= -\frac{1}{8\pi}\left\{2\frac{Q^2}{r^4} H_1\right\} Y_{l0} \ , \\
\tilde{T}_{02} &= \frac{e^{\nu}}{8\pi}\left\{2\frac{Q}{r^2}\tilde{f}_{12}\right\}\frac{\partial Y_{l0}}{\partial\theta} \ , \\
T_{00}^{\rm part} &= \frac{1}{16\pi} A_{00} Y_{l0} \ , \qquad J^0_{\rm part} = v Y_{l0} \ , \\
\tilde{F}^{0\nu}{}_{;\nu} &= -\left\{\tilde{f}_{01}{}' + \frac{2}{r}\tilde{f}_{01} - \frac{l\,(l+1)\,e^{-\nu}\tilde{f}_{02}}{r^2} - \frac{Q}{r^2}K'\right\} Y_{l0} \ , \\
\tilde{F}^{2\nu}{}_{;\nu} &= -\frac{e^{\nu}}{r^2}\left\{\tilde{f}_{12}{}' + \nu'\tilde{f}_{12}\right\}\frac{\partial Y_{l0}}{\partial\theta} \ , \\
{}^*\tilde{F}^{3\nu}{}_{;\nu} &= \frac{1}{r^2\sin\theta}\left\{\tilde{f}_{01} - \tilde{f}_{02}{}'\right\}\frac{\partial Y_{l0}}{\partial\theta} \ ,
\end{aligned}
$$

equations for $l \geq 2$:

$$
\begin{aligned}
0 &= e^{2\nu}\left[2K'' - \frac{2}{r}W' + \left(\nu' + \frac{6}{r}\right)K' - 4\left(\frac{1}{r^2} + \frac{\nu'}{r}\right)W\right] \\
&\quad - \frac{2\lambda e^{\nu}}{r^2}(W + K) - 2\frac{Q^2 e^{\nu} W}{r^4} - 4\frac{Q e^{\nu}\tilde{f}_{01}}{r^2} + A_{00}
\end{aligned}
$$

$$
0 = \frac{2}{r}W' - \left(\nu' + \frac{2}{r}\right)K' - \frac{2\lambda e^{-\nu}}{r^2}(W - K) - 2\frac{Q^2 e^{-\nu} W}{r^4} + 4\frac{Q e^{-\nu}\tilde{f}_{01}}{r^2}
$$

$$
\begin{aligned}
0 &= K'' + \left(\nu' + \frac{2}{r}\right)K' - W'' - 2\left(\nu' + \frac{1}{r}\right)W' \\
&\quad + \left(\nu'' + \nu'^2 + \frac{2\nu'}{r}\right)(K - W) - 2\frac{Q^2 e^{-\nu} K}{r^4} + \frac{4Q e^{-\nu}}{r^2}\tilde{f}_{01}
\end{aligned}
$$

$$
0 = -W' + K' - \nu' W + 4\frac{Q e^{-\nu}\tilde{f}_{02}}{r^2}
$$

$$
0 = \tilde{f}_{01}{}' + \frac{2}{r}\tilde{f}_{01} - \frac{l(l+1)e^{-\nu}\tilde{f}_{02}}{r^2} - \frac{Q}{r^2}K' + 4\pi v
$$

$$
0 = \tilde{f}_{01} - \tilde{f}_{02}{}'
$$

$$
H_0 = H_2 \equiv W, \qquad H_1 \equiv 0, \qquad \tilde{f}_{12} \equiv 0
$$

equilibrium condition (compatibility of the system):

$$m = qQ\frac{bf(b)^{1/2}}{\mathcal{M}b - Q^2}$$

coinciding with the classical condition descending from the equation of motion of the particle itself in a given RN background, i.e. $(U^\alpha = f(r)^{-1/2}\delta_0^\alpha)$

$$mU^\alpha \nabla_\alpha U^\beta = qF^\beta{}_\mu U^\mu$$

equilibrium positions:

- if $|Q| \neq \mathcal{M}$, they are separation-dependent, and require either $q^2 < m^2$ and $Q^2 > \mathcal{M}^2$ or $q^2 > m^2$ and $Q^2 < \mathcal{M}^2$;
- if $|Q| = \mathcal{M}$, we must have also $|q| = m$, so that equilibrium can occur at arbitrary separations (in agreement with the Majumdar-Papapetrou exact solution);

Reconstruction of the solution for all values of l

perturbed metric:

$$d\tilde{s}^2 = -[1-\bar{\mathcal{H}}-k(r)]f(r)dt^2+[1+\bar{\mathcal{H}}+k(r)]f(r)^{-1}dr^2+r^2[1+\bar{\mathcal{H}}](d\theta^2+\sin^2\theta d\phi^2)$$

$$k(r) = \frac{\bar{\mathcal{H}}_0 Q^2}{r^2 f(r)} \; ,$$

$$\bar{\mathcal{H}} = 2\frac{m}{br}f(b)^{-1/2}\frac{(r-\mathcal{M})(b-\mathcal{M})-\Gamma^2\cos\theta}{\bar{\mathcal{D}}}$$

$$\bar{\mathcal{D}} = [(r-\mathcal{M})^2+(b-\mathcal{M})^2-2(r-\mathcal{M})(b-\mathcal{M})\cos\theta-\Gamma^2\sin^2\theta]^{1/2}$$

$$\bar{\mathcal{H}}_0 = -2q\Gamma^2/[Q(\mathcal{M}b-Q^2)]$$

perturbed e.m. field:

$$\tilde{F} = -\left[\frac{Q}{r^2} + E_r\right] dt \wedge dr - E_\theta dt \wedge d\theta$$

with

$$\begin{aligned} E_r = -f_{01} &= \frac{q}{r^3}\frac{\mathcal{M}r - Q^2}{\mathcal{M}b - Q^2}\frac{1}{\bar{\mathcal{D}}}\Bigg\{ -\Bigg[\mathcal{M}(b-\mathcal{M}) + \Gamma^2\cos\theta \\ &\quad +[(r-\mathcal{M})(b-\mathcal{M}) - \Gamma^2\cos\theta]\frac{Q^2}{\mathcal{M}r - Q^2}\Bigg] \\ &\quad + \frac{r[(r-\mathcal{M})(b-\mathcal{M}) - \Gamma^2\cos\theta][(r-\mathcal{M}) - (b-\mathcal{M})\cos\theta]}{\bar{\mathcal{D}}^2}\Bigg\} \end{aligned}$$

$$E_\theta = -f_{02} = q\frac{\mathcal{M}r - Q^2}{\mathcal{M}b - Q^2}\frac{b^2 f(b) f(r)}{\bar{\mathcal{D}}^3}\sin\theta$$

Electric field lines (1)

Definition 1: integral curves of the differential equation

$$\frac{dx^\alpha}{d\lambda} = E(U)^\alpha$$

where $E(U)^\alpha = \tilde{F}^\alpha{}_\beta U^\beta$ is the electric field associated with

an observer with 4-velocity U;

static observers: $U = \frac{1}{\sqrt{-\tilde{g}_{tt}}}\partial_t = f(r)^{-1/2}\left(1 + \frac{\bar{\mathcal{H}} + k(r)}{2}\right)\partial_t$

→ $\frac{dr}{d\lambda} = E(U)^r \ , \qquad \frac{d\theta}{d\lambda} = E(U)^\theta$

or $-E(U)^r \, d\theta + E(U)^\theta \, dr = 0$

Electric field lines (2)

Definition 2: lines of constant flux $d\Phi = 0$

Gauss' theorem:

$$\Phi = \int_S {}^*\tilde{F} \wedge dS = 4\pi[Q + q\vartheta(r-b)] \equiv \Phi^{(0)} + \Phi^{(1)}$$

elementary flux across an infinitesimal closed surface, limited by the two spherical caps $\phi \in [0, 2\pi],\, \theta = \theta_0$, $r = r_0$ and $\phi \in [0, 2\pi],\, \theta = \theta_0 + d\theta$ and $r = r_0 + dr$:

$$d\Phi = 2\pi[{}^*\tilde{F}_{r\phi} dr + {}^*\tilde{F}_{\theta\phi} d\theta]$$

$$ {}^*\tilde{F}_{\theta\phi} = -r^2 \sin\theta \left[-(1+\bar{\mathcal{H}}) \frac{Q}{r^2} + f_{tr} \right] \equiv {}^*\tilde{F}^{(0)}_{\theta\phi} + {}^*\tilde{F}^{(1)}_{\theta\phi} \,,$$

$$ {}^*\tilde{F}_{r\phi} = f(r)^{-1} \sin\theta f_{t\theta} \equiv {}^*\tilde{F}^{(1)}_{r\phi} \,,$$

lines of constant flux: are defined as those curves solutions of the equation

$$0 = {}^*\tilde{F}_{r\phi} dr + {}^*\tilde{F}_{\theta\phi} d\theta$$

static spacetime + static family of observers
→ constant flux lines coincide with electric lines of force

in fact:

$${}^*\tilde{F}_{\theta\phi} = -\frac{\sqrt{-\tilde{g}}}{U_0} E(U)^r \ , \qquad {}^*\tilde{F}_{r\phi} = \frac{\sqrt{-\tilde{g}}}{U_0} E(U)^\theta$$

implying that

$$-E(U)^r \, d\theta + E(U)^\theta \, dr = 0$$

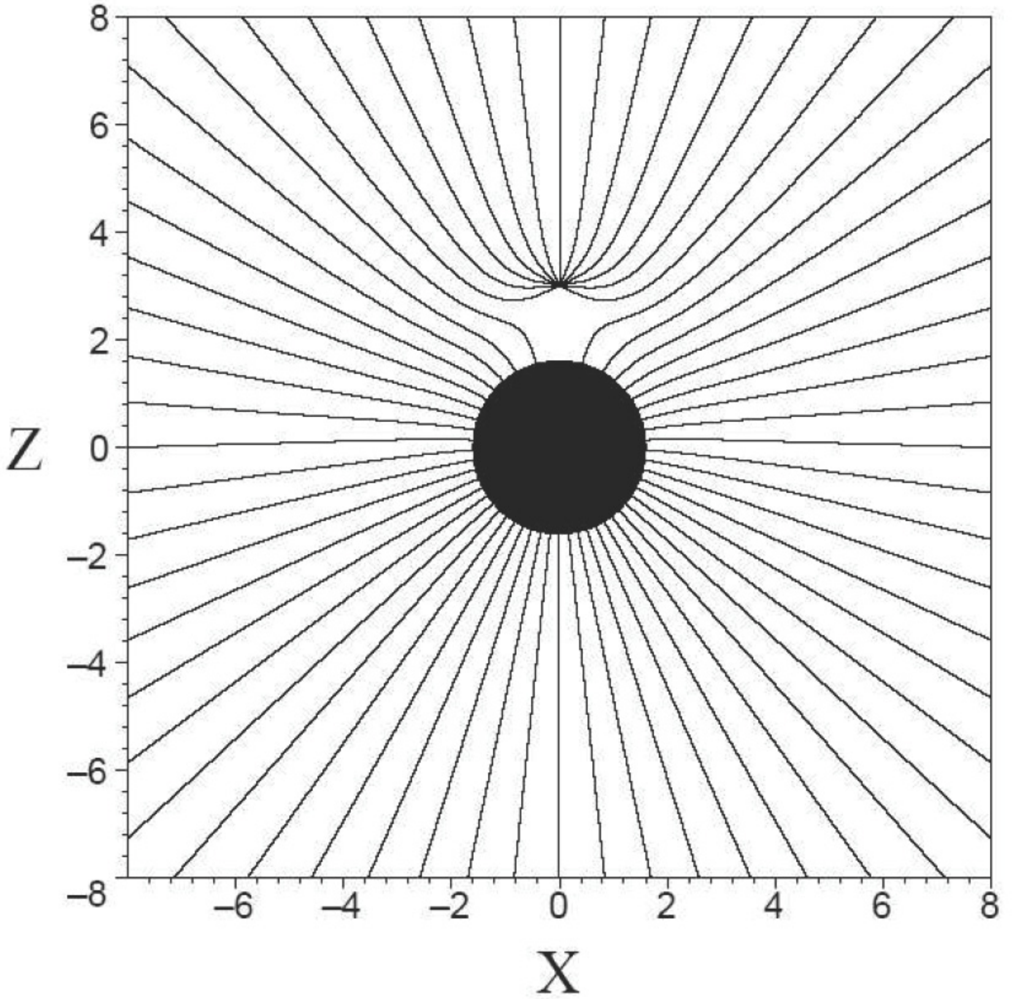
<u>Def. 1</u>:

the observer is no more a physical observer at the horizon, so that the components of the electric fields cannot be determined there

<u>Def. 2</u>:

the flux equation is well defined all the way to the horizon lines of constant flux (total field):

$$d\Phi = 0 \quad \rightarrow \quad 0 = {}^*\tilde{F}_{r\phi}dr + {}^*\tilde{F}_{\theta\phi}d\theta$$

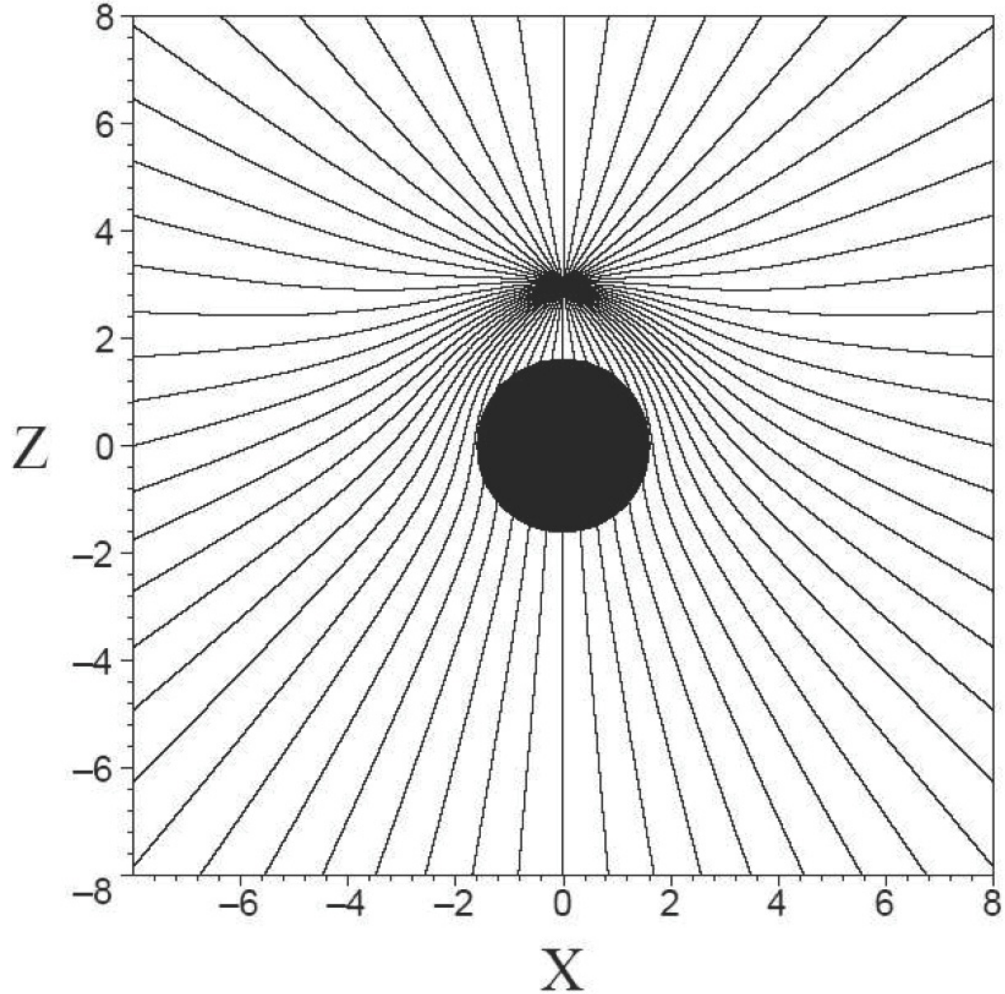


$$X = r\sin\theta$$

$$Z = r\cos\theta$$

lines of constant flux ("effective field," the BH contribution being subtracted):

$$d\Phi^{(1)} = 0 \quad \rightarrow \quad 0 = {}^*\tilde{F}^{(1)}_{r\phi}\, dr + {}^*\tilde{F}^{(1)}_{\theta\phi}\, d\theta$$



Extremely charged holes and the "electric Meissner effect"

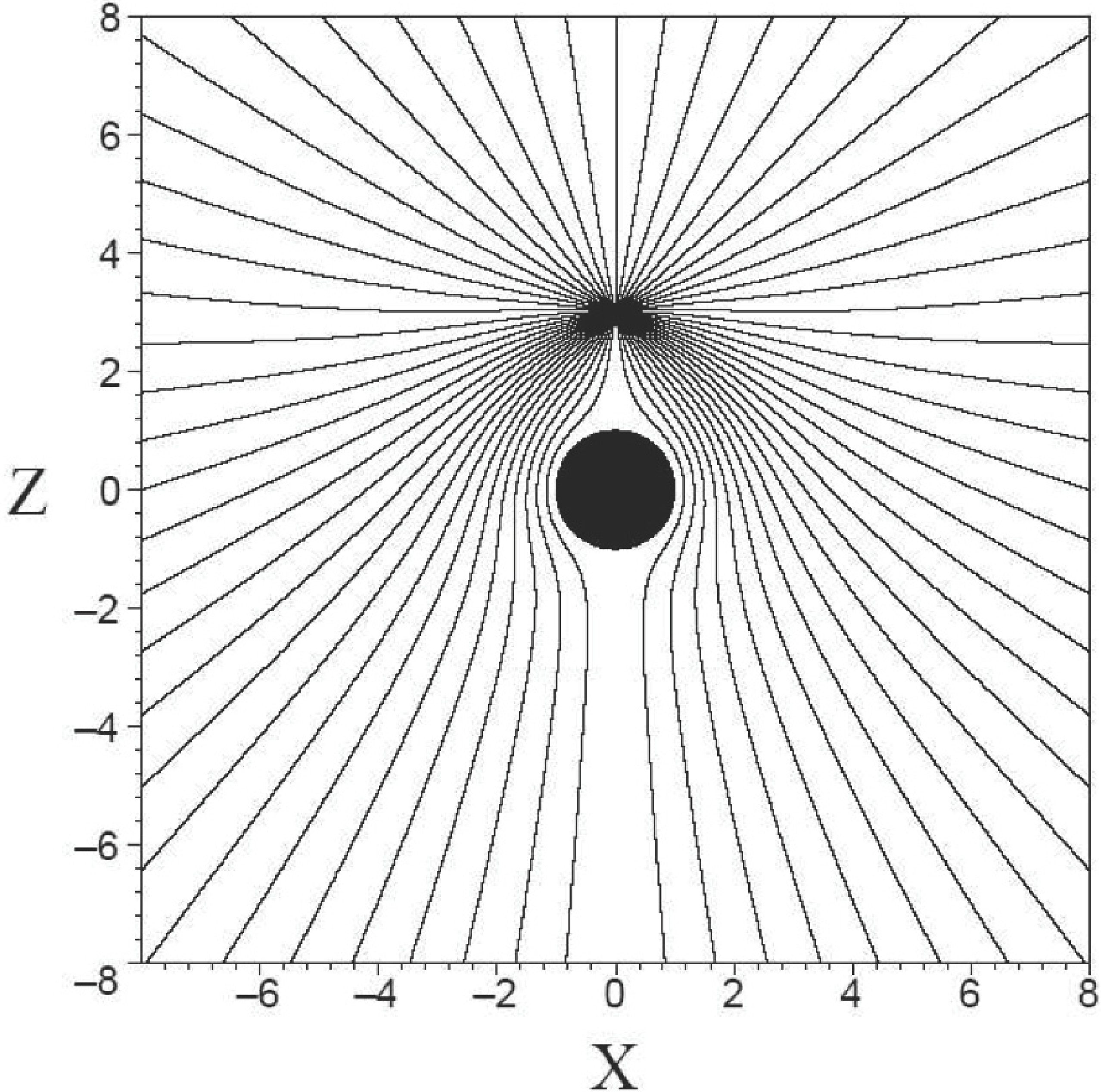
Gauss' law (surface version):

$$\frac{{}^*\tilde{F}^{(1)}_{\theta\phi}\Big|_{r_+}}{r_+^2 \sin\theta} = 4\pi\sigma^H(\theta)$$

induced charge density on the horizon and critical angle:

$$\sigma^H(\theta) = \frac{q}{4\pi r_+}\frac{\Gamma^2}{\mathcal{M}b - Q^2}\frac{\Gamma(1+\cos^2\theta) - 2(b-\mathcal{M})\cos\theta}{[b-\mathcal{M}-\Gamma\cos\theta]^2}\,,$$

$$\theta_{(\mathrm{crit})} = \arccos\left[\frac{b-\mathcal{M}-\sqrt{(b-\mathcal{M})^2-\Gamma^2}}{\Gamma}\right]\,.$$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