Almost periodic localized states: oscillons and oscillatons

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Introduction

Massive fields are necessary:

$\omega < m$ modes are non-radiative

massless fields may be present:

coupling to electromagnetism or gravitation

bosonic sector of the standard model

N. Graham *Phys. Rev.Lett.* **98**, 101801 (2007)

Nonlinearities are essential:

massive Klein-Gordon field decays as $t^{-3/2}$

a large class of finite energy initial data evolve into spherically symmetric localized oscillating configurations

Scalar field on $1 + D$ dimensional flat background

Action: $A = \int dt\, d^D x \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\partial_i \phi)^2 - U(\phi)\right]$

non-linear wave equation

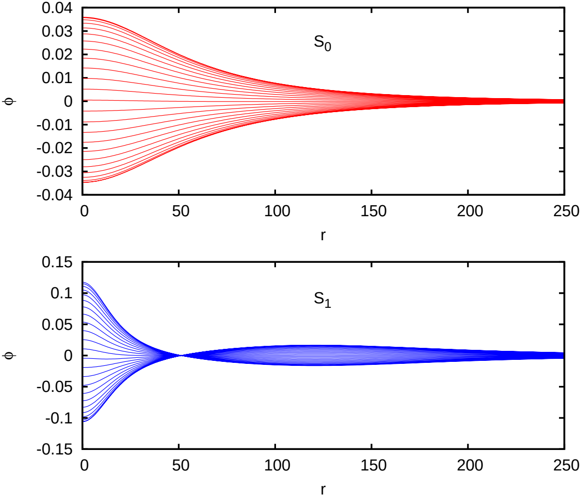
$$-\phi_{,tt} + \Delta\phi = U'(\phi)$$

minimum of potential is at $\phi = 0$, $U'(0) = 0$
$\longrightarrow$ ϕ tend to zero at infinity

mass of small excitations around $\phi = 0$ is given by $U''(0) = m^2$

rescaling t and the spatial coordinates by a constant we set $m = 1$

Shape of a typical spherically symmetric oscillon



solutions with nodes are higher energy and less stable

Scalar field history

Long lived oscillating lumps found numerically for various $U(\phi)$ by A. E. Kudryavtsev, I. L. Bogolyubskii, and V. G. Makhan'kov (JETP Letters 1975-77) called pulsons

M. Gleiser (Phys. Rev. D 1994) ϕ^4 theory, (re)named oscillons

E. J. Copeland, M. Gleiser and H.-R. Müller (PRD 1995)

$1+1$ or $1+2$ dimensions: infinitely many oscillations

$1+3$ dimensions: sudden decay after about 1000 oscillations

Massive Klein-Gordon field coupled to gravity

no need for potential term – gravity provides the nonlinearity

$$G_{ab} = \kappa T_{ab} \ , \quad T_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2} g_{ab} \left(\phi^{,c}\phi_{,c} + m^2\phi^2 \right)$$

we set

- $\kappa = 1$ by rescaling ϕ
- $m = 1$ by rescaling t and x^i

Apparently periodic oscillating lumps, found by
E. Seidel and W-M. Suen, *Phys. Rev. Lett.* **66,** 1659 (1991)

about 10 papers since then (L. A. Urena-Lopez ...)

they are called oscillatons now

metric components of oscillatons are also oscillating

- in case of boson stars the metric is static
  - fields are quantized or complex
  - R. Ruffini and S. Bonazzola,
    Phys. Rev. **187,** 1767 (1969)
  - more than 200 papers about boson stars since then

Don N. Page, *Phys. Rev. D* **70,** 023002 (2004)

- oscillatons radiate energy slowly by emitting scalar waves
- $\longrightarrow$ they are not exactly periodic

1+1 dimensional sine-Gordon theory

sine-Gordon potential $U(\phi) = 1 - \cos\phi$, $\quad -\phi_{,tt} + \phi_{,xx} = \sin\phi$
exponentially localized periodic breather solution

$$\phi = 4\arctan\left[\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\,\frac{\sin\left(\sqrt{1-\varepsilon^2}\,t\right)}{cosh\,(\varepsilon x)}\right]$$

solutions parametrized by frequency $\omega = \sqrt{1-\varepsilon^2} < 1 \quad (m = 1)$

for small ε

– amplitude is proportional to ε
– characteristic size is proportional to $\frac{1}{\varepsilon}$

these properties remain valid for any oscillons

for oscillatons amplitude is proportional to ε^2

Scalar theory with arbitrary potential

sine-Gordon potential is the only one in 1+1 dimension for which exactly periodic localized breather solutions exist

for other continuous potentials finite energy initial data evolves into long living oscillating lumps – oscillons
– they lose energy by emitting small amplitude scalar waves
– their amplitude and frequency is changing very slowly
– for small amplitudes this change is so slow that it cannot be observed numerically

in $1 + D$ dimensions ($D \geq 2$)

even sine-Gordon oscillons (pulsons) are not exactly periodic

for given D exact breather is believed to exist only for one (unknown) potential

Exactly periodic solutions

Time dependent oscillons can only be investigated by numerical evolution codes
– difficult, because massive fields produce high frequency oscillations moving out slower than the speed of light

Exactly periodic but non-localized solutions are easier to analyze – they can be calculated by solving the (truncated) Fourier mode equations numerically

$$\phi = \sum_{n=0}^{\infty} \phi_n \cos(n\,\omega t)$$

no $\sin(n\,\omega t)$ terms because of time reflection symmetry at $t = 0$

for spherical symmetry there are coupled ordinary differential equations for $\phi_n(r)$

Mode decomposition

for large radius the modes decouple ($m = 1$ was set)

$$\Delta\phi_n = (1 - n^2\omega^2)\phi_n \quad , \qquad \Delta = \frac{d^2}{dr^2} + \frac{D-1}{r}\frac{d}{dr}$$

asymptotically, ϕ_n with $n \geq n_0$ are oscillating in space
– if $\frac{1}{2} < \omega < 1$ then $n_0 = 2$

in order to have finite energy, the amplitudes of all these modes have to be zero

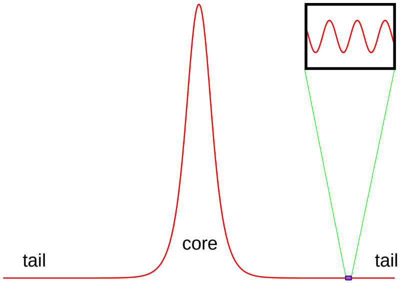
too many conditions to satisfy – generically, finite energy localized breathers are not expected to exist

Exactly periodic but only weakly localized solutions

energy loss of oscillons is compensated by small incoming radiation in order to make the solution exactly periodic

there are periodic solutions with very low energy density radiative tails

A given frequency exactly periodic solution with with the minimal energy density tail is named quasibreather



Quasibreathers

Quasibreathers agree to a high precision with the same frequency oscillon in the core region and in part of the radiative region

oscillons evolve adiabatically through quasibreather states

Total energy of quasibreathers is infinite – energy density of tail is small, but have to integrate it to infinite volume

Energy of core $E(\omega)$ is a well defined function of the frequency (core ends where spatially oscillating modes start to dominate)

For gravitational case precisely periodic configuration have infinite-mass, and spactime is not asymptotically flat

Small amplitude expansion of scalar theory

$1+1$ dimension: S. Kichenassamy,
Comm. Pur. Appl. Math. **44**, 789 (1991)

$$-\phi_{,tt} + \Delta\phi = U'(\phi) = \phi + \sum_{k=2}^{\infty} g_k \phi^k \,, \quad \text{where} \quad \Delta = \sum_{i=1}^{D} \frac{\partial^2}{\partial x_i^2}$$

for the standard ϕ^4 theory $U(\phi) = \frac{1}{8}\phi^2(\phi-2)^2$,
$g_2 = -\frac{3}{2}$, $g_3 = \frac{1}{2}$ and $g_i = 0$ for $i \geq 4$
for the sine-Gordon potential $U(\phi) = 1 - \cos(\phi)$,
$g_{2i} = 0$ and $g_{2i+1} = (-1)^i/(2i+1)!$.

Expand ϕ in terms of a small parameter ε as

$$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k$$

Rescaled coordinates

Asymptotic behaviour of the leading mode ϕ_1 shows that the size of configurations increases for small ε
We introduce rescaled spatial coordinates by

$$\zeta^i = \varepsilon x^i$$

One must also allow for the ε dependence of the time-scale of the configurations, therefore a new time coordinate is introduced as

$$\tau = \omega(\varepsilon) t$$

It is possible to show that one can always set $\omega = \sqrt{1 - \varepsilon^2}$
this corresponds to a freedom in the choice of the parametrization ε

Order ε

After these rescalings the field equation takes the form

$$-\omega^2\ddot{\phi} + \varepsilon^2 \Delta\phi = \phi + \sum_{k=2}^{\infty} g_k \phi^k$$

– overdot means $\frac{\partial}{\partial\tau}$
– spatial derivatives are calculated with respect to ζ^i

Substituting $\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k$, the ε order terms give

$$\ddot{\phi}_1 + \phi_1 = 0$$

$\phi_1 = p_1 \cos(\tau + \alpha)$, where p_1 and α are functions of ζ^i

p_1 and α will be determined at higher order

harmonic oscillator with frequency $\omega = 1$, fixed by $m = 1$

Order ε^2

The ε^2 order terms give

$$\ddot{\phi}_2 + \phi_2 + g_2\phi_1^2 = 0\ , \quad \phi_1 = p_1 \cos(\tau + \alpha)$$

the solution is

$$\phi_2 = p_2 \cos(\tau + \alpha) + q_2 \sin(\tau + \alpha) + \frac{g_2}{6} p_1^2 [\cos(2\tau + 2\alpha) - 3]$$

where p_2 and q_2 are some functions of ζ^i

Order ε^3

The ε^3 terms give another forced oscillator equation

$$
\begin{aligned}
&\ddot{\phi}_3 + \phi_3 + (p_1 \Delta\alpha + 2\nabla\alpha\nabla p_1)\sin(\tau+\alpha) \\
&-\left[\Delta p_1 - p_1 + \lambda p_1^3 - p_1(\nabla\alpha)^2\right]\cos(\tau+\alpha) \\
&+\frac{1}{12} p_1^3 (2g_2^2 + 3g_3)\cos(3\tau + 3\alpha) \\
&+g_2 p_1 \left[q_2 \sin(2\tau+2\alpha) + p_2\cos(2\tau+2\alpha) + p_2\right] = 0
\end{aligned}
$$

where $\lambda = \frac{5}{6}g_2^2 - \frac{3}{4}g_3$ has been introduced.

for ϕ^4 theory $\lambda = 3/2$, for the sine-Gordon potential $\lambda = 1/8$

the coefficients of $\sin(\tau+\alpha)$ and $\cos(\tau+\alpha)$ must vanish in order to get solutions bounded in time (resonance terms)

periodicity is a consequence of boundedness

Constant phase

The vanishing of the coefficient of the $\sin(\tau + \alpha)$ term implies

$$\nabla(p_1^2 \nabla\alpha) = 0$$

from this

$$\int_\Omega \alpha \nabla(p_1^2 \nabla\alpha) = \int_{\partial\Omega} \alpha p_1^2 n \cdot \nabla\alpha - \int_\Omega p_1^2 (\nabla\alpha)^2 = 0$$

boundary term vanishes sufficiently fast $\longrightarrow \quad \nabla\alpha = 0$
α must be a constant which can be absorbed by a shift in τ
from now on we set $\alpha = 0$
phase of oscillations is location independent

Master equation

The vanishing of the $\cos\tau$ resonance term implies

$$\Delta p_1 - p_1 + \lambda p_1^3 = 0$$

multiplying by $p1$ and integrating $\longrightarrow \quad \lambda > 0$
defining $S = p_1\sqrt{\lambda}$, where $\lambda = \frac{5}{6}g_2^2 - \frac{3}{4}g_3$ we get the master equation

$$\Delta S - S + S^3 = 0$$

– universal for the class of theories considered
– dependence on $U(\phi)$ enters only through λ

to lowest order

$$\phi = \varepsilon\phi_1 = \varepsilon p_1(\zeta^i)\cos(\tau) = \frac{\varepsilon S(\varepsilon x^i)}{\sqrt{\lambda}}\cos(\omega t)$$

where $\omega = \sqrt{1-\varepsilon^2}$

Higher orders

$\phi = \sum_{k=1}^{\infty} \varepsilon^k \phi_k$, where

$$\begin{aligned} \phi_1 &= p_1 \cos\tau \\ \phi_2 &= \frac{1}{6} g_2 p_1^2 (\cos(2\tau) - 3) \\ \phi_3 &= p_3 \cos\tau + \frac{1}{72}(4g_2^2 - 3\lambda) p_1^3 \cos(3\tau) \end{aligned}$$

– can be continued, but expressions become longer
– no $\sin(k\tau)$ terms $\longrightarrow$ time reflection symmetry
– odd ϕ_i contains only odd Fourier modes, even only even
– p_3 is determined by a linear differential equation containing source terms nonlinear in p_1
– for potentials symmetric around the minimum (sine-Gordon) $g_{2i} = 0 \longrightarrow \phi_{2i} = 0$, only odd Fourier modes

Asymptotic series

If exponentially decreasing S exist, all ϕ_n are also exponentially localized

The series solution in powers of ε does not converge to a breather, it is an asymptotic series
– H. Segur and M. D. Kruskal, *Phys. Rev. Lett.* **58**, 747 (1987) for $1 + 1$ dimension

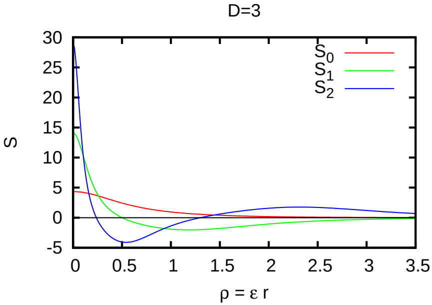
It corresponds to a quasibreather whose standing wave tail is smaller than ε^n for any $n > 0$ (beyond all orders)
tail amplitude is exponentially small $\sim \exp(-C/\varepsilon)$

To a given order in the expansion, for sufficiently small values of ε the corresponding sum yields an excellent approximation to the core part of an oscillon.

Solutions of the master equation

It can be shown that no localized solution of $\Delta S - S + S^3 = 0$ exist in dimensions $D \geq 4$

In case of spherical symmetry a discrete family of localized solutions exist for $1 < D < 4$, indexed by the number of nodes



solutions with nodes correspond to higher energy less stable oscillons

Small amplitude expansion of oscillatons

$$G_{ab} = \phi_{,a}\phi_{,b} - \frac{1}{2}g_{ab}\left(\phi^{,c}\phi_{,c} + \phi^2\right)$$

isotropic coordinates

$$ds^2 = -B(t,r)dt^2 + \frac{W(t,r)}{B(t,r)}\left(dr^2 + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2\right)$$

similar ε expansion can be made, let $\rho = \varepsilon r$

$$\phi = \sum_{n=1}^{\infty}\varepsilon^{2n}\phi_{2n}\ ,\quad B = 1 + \sum_{n=1}^{\infty}\varepsilon^{2n}B_{2n}\ ,\quad W = 1 + \sum_{n=1}^{\infty}\varepsilon^{2n}W_{2n}$$

it turns out that to leading order metric is static

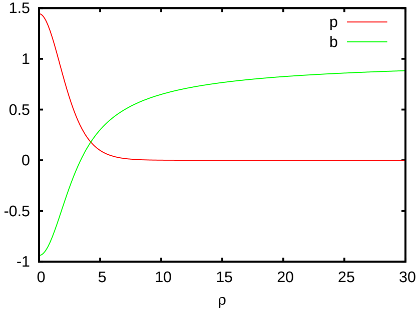
$$\phi_2(t,\rho) = p(\rho)\cos(t)\ ,\quad B_2(t,\rho) = b(\rho)\ ,\quad W_2(t,\rho) = 0$$

Schrödinger-Newton equations

master equation corresponds to Schrödinger-Newton equations

$$(\rho p)'' = \rho b p \quad , \quad (\rho b)'' = \frac{1}{2}\rho p^2$$

SN equations also describe the Newtonian limit of boson stars



Conclusions

Nonlinear massive fields may evolve into almost periodic localized configurations

Small amplitude expansion yield very good approximation to the core region

Equations are just coupled ordinary differential equations

Radiative tail is exponentially small in terms of the amplitude parameter