The Role of Time Gauge in Quantizing Gravity

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Summary

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Introduction

Loop Quantum Gravity is the only non-perturbative approach to the quantization of gravity.

It is based on recognizing Gauss-like constraints and on using techniques proper of lattice gauge theories.

Its main achievement is the emergence of a discrete spatial structure.

Nevertheless, the behavior of this structure under boosts cannot be investigated in the standard framework, since the underlying Hamiltonian formulation is based on fixing the time gauge.

ADM splitting

Let us consider an hyperbolic space-time manifold V endowed with a metric $g_{\mu\nu}$ and a $3+1$ representation

$$V \rightarrow \Sigma \otimes R,$$

Σ spatial 3-hypersurfaces $x^i \;\; (i = 1, 2, 3)$ coordinates
t real (time-like) coordinate.

vector field normal to Σ

$$n = \tilde{N}dt + \tilde{N}_i dx^i$$

$\tilde{N}$ lapse function $\tilde{N}_i$ shift vector

$$g_{\mu\nu} = \begin{pmatrix} \tilde{N}^2 - \tilde{N}^i \tilde{N}_i & -\tilde{N}_i \\ -\tilde{N}_j & -h_{ij} \end{pmatrix}$$

4-bein vectors: $e^0_\mu = (N, \chi_a E^a_i)$ $\qquad e^a_\mu = (E^a_i N^i, E^a_i)$

$$\tilde{N} = \frac{1}{\sqrt{1+\chi^2}}(N - N^i E^a_i \chi_a)$$

$$\tilde{N}^i = N^i + \frac{E^c_l \chi_c N^l - N}{1+\chi^2} E^i_a \chi^a \quad \chi^a = \chi_b \eta^{ab}$$

$$h_{ij} = -E^a_i E^b_j T^{-1}_{ab}, \; T^{-1}_{ab} = \eta_{ab} + \chi_a \chi_b$$

$$E'^a_i = E^b_i(\delta^a_b - \alpha \chi^a \chi_b) \qquad \alpha = \frac{1-\sqrt{1+\chi^2}}{\chi^2}$$

The time-gauge condition means setting e_a on Σ and it can be obtained by a boost

$$e'^B = \Lambda^B{}_A e^A$$

with $\chi_a = -\Lambda^0{}_a / \Lambda^0{}_0$.

Therefore the e^A frame is not at rest with respect to Σ, but moves with velocity components χ^a.

2^{nd} ORDER FORMULATION

configuration variables: $\tilde{N}, \tilde{N}^i, E_i^a, \chi_a$

Lagrangian density $\Lambda = \frac{1}{16\pi G}\tilde{N}\sqrt{h}(K^2 - K_{ij}K^{ij} + {}^3R)$

extrinsic curvature $K_{ij} = \frac{1}{2\tilde{N}}(D_i\tilde{N}_j + D_j\tilde{N}_i - \partial_t h_{ij})$

conjugate momenta:

$\pi_{\tilde{N}} = 0, \qquad \pi_i = 0$ Primary constraints

$$\pi_a^i = \frac{1}{8\pi G}\sqrt{h}[K^{ij}E_j^b(\delta_{ab} - \chi_a\chi_b) - KE_a^i]$$

$$\pi^a = \frac{1}{8\pi G}\sqrt{h}\left(\frac{\chi^a}{1-\chi^2}K - K^{ij}E_i^aE_j^b\chi_b\right)$$

By introducing Lagrangian multipliers $\lambda^{\tilde{N}}, \lambda^i, \lambda_a, \eta_a$ and $\tilde{N}' = \sqrt{h}\tilde{N}$, the Hamiltonian density is

$$\mathcal{H} = \tilde{N}' H + \tilde{N}^i H_i + \lambda^{\tilde{N}} \pi_{\tilde{N}} + \lambda^i \pi_i + \eta_a \varphi^a + \lambda_a \Phi^a$$

Constraints:

$$H = \pi^i_a \pi^j_b \left(\frac{1}{2} E^a_i E^b_j - E^b_i E^a_j \right) + h^3 R = 0$$ super-Hamiltonian

$$H_i = D_j(\pi^j_a E^a_i) = 0$$ super-momentum

$$\Phi^a = T^{-1ab} \pi_b + \eta^{ab} \pi^i_b \chi_c E^c_i = 0$$ boost constraints

$$\varphi^a = \epsilon^a{}_b{}^c (\pi^b \chi_c - \pi^i_c E^b_i) = 0$$ rotation constraints

The full set of constraints is of first-class.

$$\{\Phi^a; H\} \approx \{\varphi^a; H\} \approx 0 \qquad \{\Phi^a; H_i\} \approx \{\varphi^a; H_i\} \approx 0$$

Gauge fixing

By fixing $\chi_a = \bar{\chi}_a(t, x^i)$ boost constraints can be solved by

$$\pi^a = \bar{T}^{ab} \pi^i_b \bar{\chi}_c E^c_i.$$

and the action can be rewritten as

$$S = -\frac{1}{16\pi G} \int [\pi^i_a \partial_t E^a_i + \pi_{\tilde{N}'} \partial_t \tilde{N}' + \pi_i \partial_t \tilde{N}^i - \tilde{N}' H^{\bar{\chi}} - \tilde{N}^i H^{\bar{\chi}}_i - \\ - \lambda_a \varphi^a_{\bar{\chi}} - \lambda^{\tilde{N}} \pi_{\tilde{N}} - \lambda^i \pi_i] dt d^3 x,$$

where new constraints read as follows

$$\varphi^a_{\bar{\chi}} = \epsilon^{abc} (\bar{\chi}_b \pi^i_d E^d_i \bar{\chi}_d - \eta_{db} \pi^i_c E^d_i)$$
$$H^{\bar{\chi}} = H(E, \pi, \chi = \bar{\chi}),$$
$$H^{\bar{\chi}}_i = H_i(E, \pi, \chi = \bar{\chi}).$$

Transformation between $\bar{\chi}$ sectors

Let us consider the sector $\bar{\chi} = 0$, the full set of constraints reads as

$$H^0\psi_0 = 0 \quad H_i^0\psi_0 = 0 \quad \epsilon^{ab}{}_c\pi_b^i E_i^c\psi_0 = 0,$$

The action of boost constraints in the (E_i^a, π_b^j) phase space is reproduced by the operator

$$U_\epsilon = I - \frac{i}{4}\int \epsilon^a\epsilon_b(E_i^b\pi_a^i + \pi_a^i E_i^b)d^3x + O(\epsilon^4),$$

which sends 3-bein vectors for $\bar{\chi}_a = 0$ to 3-bein vectors for $\bar{\chi}_a = \epsilon_a$

As soon as an Hilbert space structure is provided, U is unitary and since

$$U_\epsilon H^0 U_\epsilon^{-1}\psi' = H^\epsilon \psi' = 0 \quad U_\epsilon H_i^0 U_\epsilon^{-1}\psi' = H_i^\epsilon \psi' = 0$$
$$U_\epsilon \epsilon^{ab}{}_c \pi_b^i E_i^c U_\epsilon^{-1}\psi' = 0 \Rightarrow \varphi_\epsilon^a \psi' = 0$$

it maps physical states between $\bar{\chi} = 0$ and $\bar{\chi} = \epsilon$ sectors.

Therefore, even though the Lorentz frame is fixed before the quantization, nevertheless the boost symmetry can be realized on a quantum level.

[Cianfrani F., Montani G., “Boost invariance of the gravitational field dynamics: quantization without time gauge”, *Classical and Quantum Gravity*, **24**, (2007) 4161-4168.]

1^{st} ORDER FORMULATION

The action of General Relativity with the Holst modification takes the following form

$$S = \frac{1}{2}\int e e^{\mu}_{A} e^{\nu}_{B} R^{CD}_{\mu\nu}(\omega^{FG}_{\mu}) {}^{\gamma}p^{AB}{}_{CD},$$

$$R^{AB}_{\mu\nu} = \partial_{[\mu}\omega^{AB}_{\nu]} + \omega^{A}_{c[\mu}\omega^{CB}_{\nu]}, \qquad {}^{\gamma}p^{AB}{}_{CD} = \delta^{AB}{}_{CD} - \frac{1}{2\gamma}\epsilon^{AB}{}_{CD}.$$

By a Legendre transformation conjugate momenta can be defined

$$ {}^{\gamma}\pi^{\mu}_{CD} = {}^{\gamma}p^{CD}{}_{AB}\pi^{\mu}_{CD}, \qquad \pi^{t}_{AB} = 0, \qquad \pi^{\mu}_{AB} = e e^{t}_{[A} e^{i}_{B]}$$

and the following constraints come out

$$\pi^{t}_{AB} = 0, \qquad C^{ij} = \epsilon^{ABCD}\pi^{(i}_{AB}\pi^{j)}_{CD} = 0,$$

Hamiltonian constraints

The Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{eg^{tt}}H - \frac{g^{ti}}{g^{tt}}H_i - \omega_t^{AB}\gamma p^{CD}{}_{AB}G_{CD} + \lambda_{ij}C^{ij} + \eta_{ij}D^{ij} + \lambda^{AB}\pi^t{}_{AB},$$

Constraint hypersurfaces:

$$H = \pi^i_{CF}\pi^{jF}{}_D\gamma p^{CD}{}_{AB}R^{AB}_{ij} = 0$$ super-Hamiltonian constraint

$$H_i = \gamma p_{AB}{}^{CD}\pi^j_{CD}R^{AB}_{ij} = 0$$ super-momentum constraints

$$G_{AB} = D_i\pi^i{}_{AB} = \partial_i\pi^i{}_{AB} - \omega_{i[A}{}^C\pi^i{}_{|C|B]} = 0$$ Gauss constraints.

$$\left.\begin{aligned} C^{ij} &= \epsilon^{ABCD}\pi^{(i}_{AB}\pi^{j)}_{CD} = 0 \\ D^{ij} &= \epsilon^{ABCD}\pi^k_{AF}\pi^{(iF}{}_B D_k\pi^{j)}_{CD} = 0 \end{aligned}\right\}$$ 2^{nd}-class constraints

Solving 2^{nd} class constraints

Geometric structures ($\pi^i_{0b} = \pi^i_b$):

$$\pi^i_a = \frac{\sqrt{h}}{\sqrt{1+\chi^2}} E^i_a, \qquad h_{ij} = \frac{1}{\pi} T^{-1}_{ab} \pi^a_i \pi^b_j$$

$$^{\pi}\omega_a{}^b{}_i = E^{b3}_j \nabla_i E^j_a = \frac{1}{\pi^{1/2}} \pi^{b3}_l \nabla_i (\pi^{1/2} \pi^l_a) \qquad (\pi^a_i \pi^i_b = \delta^a_b).$$

The solution of C^{ij} and D^{ij} is given by:

$$\omega_a{}^b{}_i = {}^{\pi}\omega_a{}^b{}_i + \chi_a \omega^{ob} + \chi^b (\omega_a{}^0{}_i - {}^{\pi}D_i \chi_a) \qquad \pi^i_{ab} = \chi_{[a} \pi^i_{b]}$$

$$({}^{\pi}D_i \chi_a = \partial_i \chi_a - {}^{\pi}\omega_a{}^b{}_i \chi_b).$$

Let us denote by Ω the corresponding hypersurfaces.

Symplectic structure:

before solving constraints

$$\{\omega_i^{AB}(x,t),\omega_j^{CD}(y,t)\}=\{\pi^i_{AB}(x,t),\pi^j_{CD}(y,t)\}=0$$

$$\{\omega_i^{AB}(x,t),\pi^j_{CD}(y,t)\}=\delta_i^j\delta^3(x-y)\,{}^{\gamma}p^{-1AB}_{CD}$$

on Ω:

$$\{\pi^i_{AB}(x,t),\pi^j_{CD}(y,t)\}=0$$

$$\begin{aligned}\{\omega_i^{a0}(x,t),\omega_j^{b0}(y,t)\}=&-\frac{1}{2\gamma(1+\chi^2)^2}T^{-1a}{}_{c}T^{-1b}{}_{d}\,\epsilon^d{}_{fg}\frac{\partial^\pi\omega_j^{fh}(y,t)}{\partial\pi^i_c(x,t)}T^{-1g}{}_{h}-\\&-\frac{1}{1+\chi^2}T^{-1a}{}_{c}\,\chi_d\frac{\partial^\pi\omega_j^{bd}(y,t)}{\partial\pi^i_c(x,t)}-(a,i\rightarrow b,j)\end{aligned}$$

$$\{\omega_i^{a0}(x,t),\pi_b^j(y,t)\}=\delta_i^j\delta^3(x-y)\frac{1}{1+\chi^2}T^{-1a}{}_{b}$$

Rotation Constraints

Rotation constraints

$$G_{ab} = T^f_{[a} T^g_{b]} \Omega_{if} \pi^i_g = 0, \quad (\Omega^a_i = \omega^{a0}_i - {}^\pi D_i \chi^a)$$

Boost constraints $G_{0a} = \chi^b \Omega_{i[a} \pi^i_{b]} = \chi^b G_{ab} = 0$

The algebra of G_{ab} turns out to be open.

The definition of a proper constraint can be made by summing the vanishing term $\partial_i \pi^i_a - {}^\pi\omega_{ia}{}^b \pi^i_b - \frac{1}{2}\partial_i \chi^2 \pi^i_a$, such that one has

$$G_a = \partial_i \pi^i_a - \frac{1}{2}\partial_i(\chi_a \chi^b) T^c_b \pi^i_c + \frac{1}{2(1+\chi^2)}\partial_i \chi^2 \pi^i_a + \gamma \epsilon_{abc} T^{cd} A^b_i \pi^i_d = 0,$$

with $$A^a_i = (1+\chi^2)T^{ab}(\omega_{b0i} - {}^\pi D\chi_b) - \frac{1}{2\gamma}\epsilon^a{}_{cd}{}^\pi\omega^{cf}{}_i T^{-1d}_{\ \ f}.$$

Constraints with a SU(2) algebra can be defined by a further redefinition of variables

$$\widetilde{\pi}^i_a = S^b_a \pi^i_b = \sqrt{h}E'^i_a, \qquad \widetilde{A}^a_i = S^{-1a}_b A^b_i,$$
$$S^a_b = \sqrt{1+\chi^2}\delta^a_b + \frac{1-\sqrt{1+\chi^2}}{\chi^2}\chi_a\chi_b$$

so having $$\{\widetilde{G}_a, \widetilde{G}_b\} = \gamma\epsilon_{ab}{}^c\widetilde{G}_c.$$

$$\widetilde{G}_a = \partial_i\widetilde{\pi}^i_a + A\partial_i\chi_a\chi^b\widetilde{\pi}^i_b + B\chi_a\partial_i\chi^b\widetilde{\pi}^i_b + C\chi_a\chi^b\partial_i\chi^2\widetilde{\pi}^i_b + \gamma\epsilon_{abc}\widetilde{A}^b_i\widetilde{\pi}^i_c = 0,$$

$$A = \frac{1-\sqrt{1+\chi^2}}{\chi^2} - \frac{1-(1-\chi^2)\sqrt{1+\chi^2}}{2(1+\chi^2)^{3/2}}, \qquad B = -\frac{1-\sqrt{1+\chi^2}}{\chi^2} - \frac{1}{2\sqrt{1+\chi^2}}, \qquad C = \ldots$$

Since

$$\{\widetilde{A}_i^a(t,x),\widetilde{\pi}_b^j(t,y)\} = \delta_b^a\delta_i^j\delta^3(x-y) \qquad \{\widetilde{A}_i^a(t,x),\widetilde{A}_j^b(t,y)\} = 0$$

and $\widetilde{\pi}_a^i$ are densitized 3-bein vectors of h_{ij}, $\widetilde{A}_i^a$ can be though as generalized Ashtekar-Barbero-Immirzi connections.

LQG in a global frame

When χ_a are constant, the Gauss constraints of a SU(2) gauge theory come out

$$\widetilde{G}_a = \partial_i \tilde{\pi}^i_a + \gamma \epsilon_{ab}{}^c \widetilde{A}^b_i \widetilde{\pi}^i_c = 0.$$

$$\widetilde{A}^a_i = T^{-1a}_b ({\omega^b}_{0i} + {}^{\pi}{\omega^{bc}}_i \chi_c) - \frac{1}{2\gamma} S^{-1a}_b {\epsilon^b}_{cd} {}^{\pi}{\omega^{cf}}_i T^{-1d}_f .$$

The LQG quantization procedure can be extended to a generic global Lorentz frame.

Other constraints coincide with those one gets when the time gauge stands.

Let us define holonomies $h(\widetilde{A})_\alpha$ and fluxes $\widetilde{\pi}(S)$

$$h(\widetilde{A})_\alpha = Pe^{\int_0^1 \widetilde{A}_i^a \dot{\alpha} dt \tau_a}$$

$$\widetilde{\pi}(S)_a = \int \widetilde{\pi}_a^i n_i du dv, \quad n_i = \epsilon_{ijk}\partial_u x^i \partial_v x^k$$

Their algebra is the same one as in LQG with the time gauge, so the quantization can be performed in the same way.

The representation on a Hilbert space is obtained by the GNS construction, taking the following ground state ω

$$\omega(f_\alpha(\widetilde{A}), \widetilde{\pi}_1, .., \widetilde{\pi}_M) = \begin{cases} \int d\mu(g_1)..d\mu(g_N) f_\alpha(g_1, .., g_N) & M = 0 \\ 0 & M \neq 0 \end{cases}.$$

This way the Hilbert space is a certain completition over the space of distributional connections

$$\mathsf{H}_{aux} = L^2(\bar{A}, d\mu),$$

A basis for this space is given by spin networks

$$\psi_S(\widetilde{A}) = \left(\prod_{v_S \in k} I_v\right) \left(\prod_{e_S \in k} D_{\rho_e} \rho_e(h_e(\widetilde{A}))\right).$$

The boost constraints can be represented on Ω by $\epsilon_a{}^{bc}\chi_b G'_c$.

The boost symmetry is actually preserved on a quantum level.

Area operator:

$$A(S) = \sqrt{1+\chi^2} \int_S \sqrt{T^{ab} S_a^{-1c} S_b^{-1d} \widetilde{\pi}^i_c \widetilde{\pi}^j_d n_i n_j} du dv$$

Let us consider a surface S and an edge e having one intersection in a point p (where the tangent to e does not belong to S). The action of the operator $A(S)$ on the parallel transport along e is given by

$$A(S) h_e(\widetilde{A}) = \gamma \sqrt{1+\chi^2} \sqrt{O} h_e(\widetilde{A}),$$
$$O = T^{ab} S_a^{-1c} S_b^{-1d} \tau_c \tau_d = \frac{1}{1+\chi^2} \eta^{ab} \tau_a \tau_b = C_j$$

C_j being the Casimir operator of the SU(2) representation.

$$A(S)h_e^j(\widetilde{A}) = \gamma\sqrt{j(j+1)}h_e^j(\widetilde{A})$$

Therefore, spin networks turn out to be eigen-states of the area operator.

The area spectrum is discrete and this discrete structure is not modified in a global Lorentz frame.

Conclusions

It has been outlined how it is possible to generalize Ashtekar-Barbero-Immirzi connections and to infer Gauss-constraints in a generic global Lorentz-frame.

This extension of LQG confirms that the spectra of geometrical operators are Lorentz-invariant (at least for global transformations).

What about a local Lorentz frame ($\chi_a \neq cons.$)?