Review on Extended Approaches in the Kaluza-Klein Model

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Summary

- Kaluza-Klein Theory
- KK models for non-Abelian theories
- Fermions in a KK framework
- Electro-Weak model
- Outlook

Kaluza-Klein theory

The original Kaluza-Klein theory was developed in 1919 by Kaluza to unify gravity and electrodynamics.

The basic idea is to add a closed spatial dimension to the space-time

$$V = V^4 \otimes S^1$$

and to interpret some of the additional components of the metric as the electromagnetic field A_μ.

$$j_{AB} = \left(\begin{array}{c|c} g_{\mu\nu} - e^2k^2A_\mu A_\nu & -ekA_\mu \\ \hline & \\ -ekA_\nu & -1 \end{array}\right) \qquad (1)$$

As soon as it is assumed that *no quantity depends on the fifth coordinate* (cilindricity condition), one has

- Abelian gauge transformation are reproduced for A_μ by translation along S^1

$$y' = y + \omega(x^\mu) \Rightarrow A'_\mu = A_\mu - \partial_\mu \omega$$

- the Maxwell Lagrangian density comes out from Einstein-Hilbert action

$$S = \frac{c^4}{16\pi G}\int_{V^4 \otimes S^1} {}^5R d^4x dy = \frac{c^4}{16\pi G}\int_{V^4}\left(R - \frac{e^2k^2}{4}F_{\mu\nu}F^{\mu\nu}\right)$$

An explanation for the cilindricity condition came from Klein:

a generic function can be Fourier expanded in y

$$f(x,y) = \sum_{n=-\infty}^{+\infty} f_n e^{iny/L},\ p_5^n \approx \frac{n}{L}.$$

In a 4-dimensional point of view, any mode appears as an independent field (KK tower).

The cilindricity condition is based on an extra-dimensional space compactified to scales well below those reached in experiments, such that all modes except the 0-one are suppressed.

An estimate for L is obtained by the introduction of matter, which must be consistent with the geometrized scheme.

The 4-dimensional reduction of a test particle trajectory gives a deviation with respect to 4-geodesics

$$ {}^{(5)}u^A \nabla_A {}^{(5)}u_B = 0. \Rightarrow \begin{cases} q = \frac{\sqrt{4G}}{c} m u_5 = const. \\ u^\nu \nabla_\nu u_\mu = \frac{q}{mc^2} F_\mu{}^\nu u_\nu \end{cases}, $$

which can be interpreted as due to an electric charge induced by a dependence on y.

Therefore the introduction of test particles is consistent with the KK geometrization procedure.

The relation between the 5-momentum $p_5 = mu_5$ and the electric charge gives, once the electron charge is taken as the minimum value for e,

$$L = 8\pi^2\sqrt{G}\frac{\hbar}{ec} \approx 1.49\ 10^{-30}cm.$$

Indeed, this formulation cannot be applied to elementary particles since one must impose $u_5^2 < 1$, hence $\frac{q}{m} < 2\sqrt{G} \sim 10^{-4}\frac{C}{kg}$ (q/m problem).

Let us now consider the next order of a multipole expansion, the pole-dipole case, i.e.

$$\begin{cases} \frac{D}{{}^{(5)}Ds}{}^{(5)}P^A = \frac{1}{2}{}^{(5)}R_{BCD}{}^{A}\Sigma^{BC\,(5)}u^D, \\ \frac{D}{{}^{(5)}Ds}\Sigma^{AB} = {}^{(5)}P^{A\,(5)}u^B - {}^{(5)}P^{B\,(5)}u^A \\ {}^{(5)}P^A = {}^{(5)}m^{(5)}u^A - \frac{D\Sigma^{AB}}{{}^{(5)}Ds}{}^{(5)}u_B, \\ \Sigma^{AB\,(5)}u_A = 0 \end{cases}$$

we identify 4-quantities

$$S^{\mu\nu} = \Sigma^{\mu\nu} \qquad S_\mu = \Sigma_{5\mu} \qquad {}^{(5)}m = \frac{1}{\sqrt{1-u_5^2}}m = \alpha m$$

By the splitting procedure we get Dixon-Souriau equations

$$\left\{\begin{array}{c} \frac{D}{Ds}\hat{P}^\mu = \frac{1}{2}R_{\alpha\beta\gamma}{}^{\mu}S^{\alpha\beta}u^\gamma + qF^\mu{}_\nu u^\nu + \frac{1}{2}\nabla^\mu F^{\nu\rho}M_{\nu\rho}. \\ \frac{DS^{\mu\nu}}{Ds} = \hat{P}^\mu u^\nu - \hat{P}^\nu u^\mu + F^\mu{}_\rho M^{\rho\nu} - F^\nu{}_\rho M^{\rho\mu} \\ \hat{P}^\mu = \alpha^2 P^\mu + u_5\frac{DS^\mu}{Ds} - ekF_{\rho\nu}u^\rho S^{\nu\mu}u_5 + \frac{1}{2}ekF^\mu{}_\rho S^\rho \end{array}\right. ,$$

The geometrization of the electro-magnetic fields does not modify the dynamics of the moving object, up to the dipole order.

[Cianfrani F., Milillo I., Montani G.,“Dixon-Souriau equations from a 5-dimensional spinning particle in a Kaluza-Klein framework”, *Phys. Lett. A*, **366**, (2007), 7.]

The following identifications stand:

- $q = \alpha^2 \widetilde{P}_5 + \frac{1}{4} ekF_{\mu\nu}S^{\mu\nu}$ electric charge
  $(\widetilde{P}_5 = mu_5 - u_\nu \frac{DS^\nu}{Ds} + ekF_{\rho\nu}u^\rho S^{\nu\mu}u_\mu)$
- $M^{\mu\nu} = \frac{1}{2} ek(S^{\mu\nu}u_5 + u^\mu S^\nu - u^\nu S^\mu)$ electro-magnetic moment:
  - $S^{\mu\nu}$ 4-spin tensor
  - S^μ electric part

Therefore, the extra-components of the 5-spin tensor $(\Sigma_{5\mu})$ describe a non-vanishing electric moment.

KK models for non-Abelian theories

The KK approach can be extended to non-Abelian gauge theories by taking a homogeneous compact extra-dimensional manifold:

$$\text{Space-Time manifold: } V^4 \otimes B^K$$

B^K endowed with Killing vectors, whose algebra reproduces the gauge group one

$$\xi^n_{\bar{N}}\partial_n\xi^m_{\bar{M}} - \xi^n_{\bar{M}}\partial_n\xi^m_{\bar{N}} = C^{\bar{P}}_{\bar{N}\bar{M}}\xi^m_{\bar{P}} \tag{2}$$

As a consequence, no cilindricity condition can be imposed.

metric tensor ($\mu, \nu = 0, .., 3 \quad m, n = 1, .., K$)

$$
j_{AB} = \left(\begin{array}{c|c} g_{\mu\nu} - \phi^2 \gamma_{mn} \xi^m_{\bar{M}} \xi^n_{\bar{N}} A^{\bar{M}}_\mu A^{\bar{N}}_\nu & -\phi^2 \gamma_{mn} \xi^m_{\bar{M}} A^{\bar{M}}_\mu \\ \hline -\phi^2 \gamma_{mn} \xi^n_{\bar{N}} A^{\bar{N}}_\nu & -\phi^2 \gamma_{mn} \end{array} \right) \tag{3}
$$

$g_{\mu\nu} = g_{\mu\nu}(x^\rho)$ four-dimensional metric

$\gamma_{mn} = \gamma_{mn}(y^r)$ extra-dimensional metric

$\phi = \phi(x^\rho)$ conformal factor

$A^{\bar{M}}_\mu = A^{\bar{M}}_\mu(x^\rho)$ GAUGE BOSONS

From Einstein-Hilbert action in 4+K dimensions, the Yang-Mills lagrangian density comes out

$$S=-\frac{c^3}{16\pi{}^{(n)}G}\int_{V^4\otimes B^K}\sqrt{-j}\,{}^{(n)}Rd^4xd^Ky$$
$$\Downarrow$$
$$S=-\frac{c^3}{16\pi G}\int_{V^4}\sqrt{-g}\phi\left[R+{}^{(K)}R'+\frac{1}{4}\phi^2F^{\bar{M}}_{\mu\nu}F^{\bar{M}\mu\nu}\right]d^4x$$

as soon as the following relations stand

$$G=\frac{{}^{(n)}G}{V^K}\qquad\int_{B^K}\sqrt{-\gamma}[\gamma_{rs}\xi^r_{\bar{M}}\xi^s_{\bar{N}}]d^ky=-V^K\delta_{\bar{M}\bar{N}}. \qquad (4)$$

However, it is highly non-trivial to reproduce non-Abelian gauge transformations on gauge bosons

$$y'^{m} = y^{m} + \omega^{\bar{N}}(x^{\nu})\xi^{m}_{\bar{N}}(y^{n})$$

$$\Downarrow$$

$$\xi'^{\ m}_{\bar{M}}(y')A'^{\bar{M}}_{\mu} = \xi'^{\ m}_{\bar{M}}(y')(A^{\bar{M}}_{\mu} - \partial_{\mu}\omega^{\bar{M}}) = \xi^{m}_{\bar{M}}(y)(A^{\bar{M}}_{\mu} + C^{\bar{M}}_{\bar{P}\bar{Q}}A^{\bar{P}}_{\mu}\omega^{\bar{Q}} - \partial_{\mu}\omega^{\bar{M}}).$$

This can be done by an averaging procedure on B^K

$$\int d^K y' \sqrt{\gamma}\xi'^{\ m}_{\bar{M}}(y')A'^{\bar{M}}_{\mu} = \int d^K y \sqrt{\gamma}\xi^{m}_{\bar{M}}(y)(A^{\bar{M}}_{\mu} + C^{\bar{M}}_{\bar{P}\bar{Q}}A^{\bar{P}}_{\mu}\omega^{\bar{Q}} - \partial_{\mu}\omega^{\bar{M}}).$$

A further issue comes from equations of motion one gets from the action before the dimensional reduction.

They contain a dependence on extra-coordinates.

$$R_{AB}(x,y)=0$$

It is possible to reconcile them with those ones coming from the action after the dimensional reduction by the averaging procedure:

$$\int{}^{(4+K)}R_{AB}\sqrt{\gamma}d^K y=0.\Rightarrow\begin{cases} G_{\mu\nu}=\frac{8\pi G}{c^4}T^A_{\mu\nu} \\ D_\nu F^{\bar{M}\mu\nu}=0 \\ F^{\bar{M}}_{\mu\nu}F^{\bar{N}\mu\nu}=const. \end{cases}$$

Equations above outline that the geometrization of the gauge boson component can be achieved from equations of motion too, as soon as an averaging procedure is defined (the last one emphasizes the role of the conformal factor).

The average is then a necessary tool to find out 4-phenomenology. Its physical meaning is related with the un-detectability of B^K.

[Cianfrani F., Montani G., "Non Abelian gauge symmetries induced by the unobservability of extra-dimensions in a Kaluza-Klein approach", *Mod. Phys. Lett.*, **A21**, 3, (2006), 265.]

Fermions in a KK framework

Multidimensional spinors has been provided by simply extending the four-dimensional formalism; this line of thinking has to face non trivial questions:

_eigenvalues of the Dirac equation on the extra-space behave like masses and they are of the compactification scale order.

$$i\gamma^{\mu}(D_{\mu} - A_{\mu}^{(m)} e_{(m)}^{m} \partial_{m})\Psi + i\gamma^{m} D_{m}\Psi - m\Psi = 0 \tag{5}$$

This fact leads to the model inconsistency because for the multidimensional spinors a non zero-eigenvalue state exists.

_different transformation properties for left-handed and right-handed spinors have to be implemented in a geometrical way.

The hope was that the right-handed and left-handed zero modes of the Dirac operator behaved differently under n-bein rotations.

This possibility is ruled out by the Atiyah-Hirzebruch theorem.

These issues can be solved by introducing external gauge bosons, thus destroying the most important result, *i.e.* the geometrization of the boson component.

Nevertheless, even with external gauge bosons a consistent Kaluza-Klein picture for the Standard Model was not developed.

Here we outline how the averaging procedure can a non-trivial effect, such that 4-spinors able to solve the above issues arise.

Example: Dirac equation on S^3

Let us consider a spinor on $V^4 \otimes S^3$.

The relic un-broken symmetry group in the full tangent space is $SO(1;3) \otimes SO(3)$ and therefore we can build up an eight-component representation in the following way:

$$^{(7)}\Psi_r = \chi_{rs}\psi_s \qquad r,s = 1,2 \tag{6}$$

$\chi = \chi(y)$ being a $SU(2)$ representation and $\psi_s = \psi_s(x)$ Dirac spinors.

Thus, we look for a solution of the following averaged Dirac equation

$$\int_{S^3} d^3y\sqrt{\gamma}\gamma^{(m)}(e^m_{(m)}\partial_m - \frac{i}{2}\sigma_{(m)})\chi = 0. \tag{7}$$

We take the following form for χ

$$\chi_{rs} = \frac{1}{\sqrt{V}} e^{-\frac{i}{2}\sigma_{(p)rs}\lambda^{(p)}_{(q)}\Theta^{(q)}(y^m)} \tag{8}$$

with V the volume of S^3, λ a constant matrix satisfying

$$(\lambda^{-1})^{(p)}_{(q)} = \frac{1}{V}\int_{S^3}\sqrt{-\gamma}e^m_{(q)}\partial_m\Theta^{(p)}d^3y \tag{9}$$

and for Θ $\qquad \Theta^{(p)} = \frac{1}{\beta} c^{(p)} e^{-\beta\eta} \qquad \eta > 0.$

By performing an expansion in β^{-1}, we find

$$\int_{S^3} d^3y \sqrt{\gamma} e^m_{(m)} \partial_m \chi = \frac{i}{2} \sigma_{(m)} \chi + O(\beta^{-1}) \tag{10}$$

thus, *the bigger the parameter β is, the better the spinor approximates the solution of the massless Dirac equation*.

This way we can control by an order parameter additional terms arising from in the extra-dimensional Dirac equation.

These terms will provide violations of gauge symmetries, whose conservation can account for its small value.

Electro-Weak model

The Electro-Weak model is developed as a $SU(2) \otimes U(1)$ gauge theory. The bosonic component is reproduce geometrically by a KK space-time $V^4 \otimes S^3 \otimes S^1$.

Since in this space-time multidimensional spinors have 16 components, they can be arranged such that any 8-spinor contains a quark generation and a fermion family, i.e.

$$
\Psi_L = \frac{1}{\sqrt{V\alpha'}} \begin{pmatrix} \chi \begin{pmatrix} e^{in_{uL}\theta} u_L \\ e^{in_{dL}\theta} d_L \end{pmatrix} \\ \chi \begin{pmatrix} e^{in_{\nu L}\theta} \nu_{eL} \\ e^{in_{eL}\theta} e_L \end{pmatrix} \end{pmatrix}
$$

$$
\Psi_R = \frac{1}{\sqrt{V\alpha'}} \begin{pmatrix} \begin{pmatrix} e^{in_{uR}\theta} u_R \\ e^{in_{dR}\theta} d_R \end{pmatrix} \\ \begin{pmatrix} e^{in_{\nu R}\theta} \nu_{eR} \\ e^{in_{eR}\theta} e_R \end{pmatrix} \end{pmatrix} .
$$

This way, *from 6 multidimensional spinors all Standard Model particles can be reproduced and an explanation for the equal number of quark generations and fermion families is provided.*

A scalar field can be introduced, which realizes the spontaneous symmetry breaking mechanism to $U(1)$ and gives masses to particles. This way we are able to

- stabilize Higgs mass, by a KK mass term;
- give mass to neutrinos.

Estimates for the parameter β come out from gauge-violating term, in particular

from

$$\Gamma(n \rightarrow p + \nu_e + \bar{\nu}_e)/\Gamma_{tot} < 8 * 10^{-27} \Rightarrow \beta > 10^{14}. \tag{11}$$

from

$$m_\gamma < 6 * 10^{-17} eV \Rightarrow \beta > 10^{28} \tag{12}$$

Outlook

Prospectives are

- a better characterization of the averaging procedure;
- the search for an explanation of the chirality;
- the introduction of strong interaction.

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