Review on the Generic Cosmological Solution Near the Singularity

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III Stueckelberg Workshop
Pescara, 17^{th} July 2008

Introduction

The relevance of the Belinski-Khalatnikov-Lifshitz (BKL) work relies on two points

1. It provided a piecewise analytical solution of the Einstein equations
2. The corresponding generic cosmological solution exhibits a chaotic behaviour.

Why believe that Universe behaviour is related to a chaotic cosmology?

1. The Standard Cosmological Model is based on the highly symmetric FRW model, and observations agree with these assumptions.
2. A correspondence between theory and data exists even for an inflationary scenario.

The validity of the BKL regime must be settled down in pre-inflationary evolution, although unaccessible to present observations.

Introduction

The relevance of the Mixmaster dynamics can be identified as follows

1. The oscillatory regime can describe the **mechanism of transition to a classical cosmology**.
2. The **FRW model is backward unstable** with respect to tensor perturbations.
3. The **inflationary scenario offers an efficient isotropization mechanism**, able to reconcile the primordial inhomogeneous Mixmaster with the local high isotropy of the sky sphere at the recombination age.
4. At Planckian scale, quantum fluctuations can be correlated at most on the causal scale, thus we should regard **global symmetries as approximated toy models.**

Outline

1. The Oscillatory Regime
   - BKL analysis of the Mixmaster
   - The Generic Cosmological solution
   - Minisuperspace Description
2. Cosmological Chaos
   - Chaotic properties of the Mixmaster: the Gauss Map and the Invariant Measure
   - Lyapunov Exponent and the long discussion on Chaos Covariance
   - Cosmological Chaos as a dimensional and matter dependent phenomenon
3. Non Classical Properties
   - WDW Equation in Misner variables
   - Semi-classical and quantum properties in Misner-Chitré variables
4. Conclusions and Bibliography

The Oscillatory Regime

Einstein Equations for a homogeneous model

Form of the 3-metric

Synchronous frame of reference, the metric is taken diagonal $ds^2 = dt^2 - h_{\alpha\beta} dx^\alpha dx^\beta$

$$h_{\alpha\beta} = a^2(t) l_\alpha l_\beta + b^2(t) m_\alpha m_\beta + c^2(t) n_\alpha n_\beta \qquad \alpha, \beta, \gamma = 1, 2, 3$$

Einstein field equation for the Bianchi models

$$-R_l^l = \frac{(\dot{a}bc)^\cdot}{abc} + \frac{1}{2a^2b^2c^2}\left[\lambda^2 a^4 - \left(\mu b^2 - \nu c^2\right)^2\right] = 0$$

$$-R_m^m = \frac{(a\dot{b}c)^\cdot}{abc} + \frac{1}{2a^2b^2c^2}\left[\mu^2 b^4 - \left(\lambda a^2 - \nu c^2\right)^2\right] = 0$$

$$-R_n^n = \frac{(ab\dot{c})^\cdot}{abc} + \frac{1}{2a^2b^2c^2}\left[\nu^2 c^4 - \left(\lambda a^2 - \mu b^2\right)^2\right] = 0$$

$$-R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0$$

$(\lambda,\ \mu,\ \nu)$ specifies the homogeneous model

- for the Bianchi VIII model: $(1,\ 1,\ -1)$
- for the Bianchi IX model: $(1,\ 1,\ 1)$

BKL analysis of the Bianchi type IX model

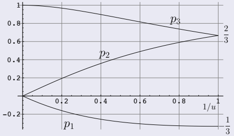
This system of equations is not analytically integrable...

...but a piece-wise solution can be constructed!

BKL construction of the solution

- Assume the potential term to be negligible at a certain t^*
  then the solution is Kasner like.

$$\boxed{\begin{gathered} a \sim t^{2p_1}, b \sim t^{2p_2}, c \sim t^{2p_3}, \\ p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1 \end{gathered}}$$



- one index is always lower than 0 (say p_1)
  so that $\boxed{a(t) \sim t^{4p_1} \longrightarrow \infty \text{ as } t \longrightarrow 0}$
  this solution is unstable backward in time!!

- If a^4 term is taken into account
  $\downarrow$
  Bianchi type II dynamics

$$\begin{cases} a^2 &= \dfrac{2|p_1|\Lambda}{\cosh(2|p_1|\Lambda\tau)} \\ b^2 &= {b_0}^2 \exp\left[2\Lambda\left(p_2 - |p_1|\right)\tau\right] \cosh\left(2|p_1|\Lambda\tau\right) \\ c^2 &= {c_0}^2 \exp\left[2\Lambda\left(p_3 - |p_1|\right)\tau\right] \cosh\left(2|p_1|\Lambda\tau\right) \end{cases}$$

BKL map

[BKL *Adv. Phys* **19**, 525 (1970).]

The perturbation induces a transition from a Kasner epoch to another

$$\tau \to \infty : \quad \begin{cases} a & \sim \exp\left[-\Lambda p_1 \tau\right] \\ b & \sim \exp\left[\Lambda\left(p_2 + 2p_1\right)\tau\right] \\ c & \sim \exp\left[\Lambda\left(p_3 + 2p_1\right)\tau\right] \\ t & \sim \exp\left[\Lambda\left(1 + 2p_1\right)\tau\right] \end{cases} \quad \Rightarrow \quad \boxed{\begin{matrix} a \sim t^{p'_l}, \quad b \sim t^{p'_m}, \quad c \sim t^{p'_n} \\ abc = \Lambda' t \end{matrix}}$$

ANOTHER KASNER EPOCH!

BKL Map

$$p'_l = \frac{|p_1|}{1 - 2|p_1|} \qquad p'_m = -\frac{2|p_1| - p_2}{1 - 2|p_1|} \qquad p'_n = \frac{p_3 - 2|p_1|}{1 - 2|p_1|} \qquad \Lambda' = (1 - 2|p_1|)\Lambda$$

- The second Kasner epoch starts with a different negative Kasner index (say p_2).
- The a^4 term starts decreasing, while b^4 starts increasing....
- ...increasing up to induce a new "transition" to another Kasner epoch

Piecewise solution

Kasner Epochs and Eras

- u parametrization:

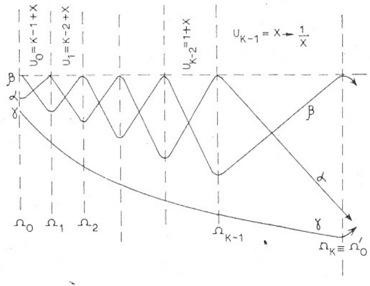
$$p_1(u) = \frac{-u}{1+u+u^2} \qquad p_2(u) = \frac{1+u}{1+u+u^2} \qquad p_3(u) = \frac{u(1+u)}{1+u+u^2}$$

$$\left.\begin{array}{l} p_l = p_1(u) \\ p_m = p_2(u) \\ p_n = p_3(u) \end{array}\right\} \Rightarrow \left\{\begin{array}{l} p'_l = p_2(u-1) \\ p'_m = p_1(u-1) \\ p'_n = p_3(u-1) \end{array}\right.$$

- The full u-map:

$$u' = u - 1 \quad \text{for} \quad u > 2$$

$$u' = \frac{1}{u-1} \quad \text{for} \quad u \leq 2$$



Formulation of the Generic cosmological problem

BKL in the 70's derived this solution and showed how its dynamics resembles the one of Bianchi types VIII and IX.

The construction can be achieved in two steps

1. firstly considering the generic solution for the individual Kasner epoch,
2. then providing a general description of the alternation of two successive epochs.

Results

1. Generalized Kasner Solution (GKS)
2. A Replacement rule for the homogeneous indices

Generalized Kasner solution

[I.M. Khalatnikov and E.M. Lifshitz. *Adv. Phys.* **12**, *185 (1963)*]

KL in 1963 showed that the Kasner solution can be generalized to the inhomogeneous case and near the singularity

$$\begin{cases} dl^2 = h_{\alpha\beta} dx^\alpha dx^\beta \\ h_{\alpha\beta} = a^2 l_\alpha l_\beta + b^2 m_\alpha m_\beta + c^2 n_\alpha n_\beta \end{cases}$$

$$a \sim t^{p_l}\,, \qquad b \sim t^{p_m}\,, \qquad c \sim t^{p_n}$$

$$\boxed{p_l(x^\gamma) + p_m(x^\gamma) + p_n(x^\gamma) = p_l^2(x^\gamma) + p_m^2(x^\gamma) + p_n^2(x^\gamma) = 1}$$

and the frame vectors l, m, n are now arbitrary functions of the coordinates

but this solution requires an additional condition to be stable

$$l \cdot \nabla \wedge l = 0$$

and this make GKS not general (not enough arbitrary functions)

Total functions =	12	(9 vector components + 3 Kasner indices)
Constraints =	9	(2 Kasner relations + 3 0α eqns + 3 conditions from the invariance under three-dimensional coordinate transformations + $l \cdot \nabla \wedge l = 0$.)
Arbitrary functions =	3	A general solution possesses 4 functions

The Generic Cosmological solution

[BKL *Adv. Phys* **31**, 639 (1982)]

Near the singularity, the matter energy-momentum tensor in the 00- and $\alpha\beta$-components may be neglected

$$-R_0^0 = \frac{1}{2}\dot{\chi}_\alpha^\alpha + \frac{1}{4}\chi_\beta^\alpha\chi_\alpha^\beta = 0$$

$$-R_\alpha^\beta = \frac{1}{2\sqrt{h}}\partial_t\left(\sqrt{h}\chi_\alpha^\beta\right) + P_\alpha^\beta = 0$$

The GKS is obtained neglecting the three-dimensional Ricci tensor P_α^β

Constraints: $P_l^l,\ P_m^m,\ P_n^n \ll t^{-2}\ , \qquad P_l^l \gg P_m^m,\ P_n^n$

Necessary and sufficient conditions: $a\sqrt{k/\Lambda} \ll 1\ , \qquad b\sqrt{k/\Lambda} \ll 1\ , \qquad c\sqrt{k/\Lambda} \ll 1$

$1/k \sim$ spatial distances over which the metric significantly changes.

As t decreases, these conditions may be violated

$$a_{tr} = \sqrt{\frac{k}{\Lambda}} \sim 1$$

The Generic Cosmological solution

[BKL *Adv. Phys* **31**, 639 (1982).]

Einstein Equations

$$-R_l^l = \frac{(\dot{a}bc)^{\cdot}}{abc} + \lambda^2 \frac{a^2}{2b^2c^2} = 0 \qquad -R_n^n = \frac{(ab\dot{c})^{\cdot}}{abc} - \lambda^2 \frac{a^2}{2b^2c^2} = 0$$

$$-R_m^m = \frac{(a\dot{b}c)^{\cdot}}{abc} - \lambda^2 \frac{a^2}{2b^2c^2} = 0 \qquad -R_0^0 = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0$$

which differ from those of the IX type only for the quantity

$$\lambda(x) = \frac{l \cdot \nabla \wedge l}{l \cdot [m \times n]}$$

no longer being a constant, but a function of the space coordinates.

Since this is a system of ordinary differential equations with respect to time where space coordinates enter parametrically only **such difference does not affect at all the solution and the map.**
Similarly, the law of alternation of exponents derived for homogeneous indices remains valid in the general inhomogeneous case.

Homogeneous Minisuperspace Models

3-Metric in vacuum $$h_{\alpha\beta} = e^{q_a}\delta_{ab}e^a_\alpha(x)e^b_\beta(x)$$

Action for a generic homogeneous model

$$S = \int dt\, \left(p_a \partial_t q^a - NH\right)$$

$$H = \frac{1}{\sqrt{\eta}}\left[\sum_a (p_a)^2 - \frac{1}{2}\left(\sum_b p_b\right)^2 - \eta\ ^{(3)}R\right]$$

$$\eta \equiv \exp \sum_a q_a$$

The anisotropy parameters and the ^{3}Ricci scalar

- Anisotropy parameters $$Q_a \equiv \frac{q^a}{\sum_b q^b}, \qquad \sum_{a=1}^{3} Q_a = 1$$
- "Potential" term $$U = \eta\ ^{(3)}R = \sum_a \lambda_a^2 \eta^{2Q_a} - \sum_{b\neq c} \lambda_b \lambda_c \eta^{Q_b+Q_c}$$
- Asymptotic behaviour of the potential $$U = \sum_a \Theta(Q_a) \qquad \Theta(x) = \begin{cases} +\infty, & \text{if } x < 0 \\ 0, & \text{if } x > 0\,. \end{cases}$$

Misner variables

[C. W. Misner *Phys Rev Lett* **22**, 1071 (1969) and *Phys Rev* **186**, 1319 (1969)]

Misner variables α, β_+, β_-

$$\eta_{ab} = e^{2\alpha}\left(e^{2\beta}\right)_{ab} \quad \Leftrightarrow \quad \begin{cases} \beta_{11} = & \beta_+ + \sqrt{3}\beta_- \\ \beta_{22} = & \beta_+ - \sqrt{3}\beta_- \\ \beta_{33} = & -2\beta_+ \end{cases}$$

Line Element

$$ds^2 = N^2(t)dt^2 - e^{2\alpha}(e^{2\beta})_{ab}\omega^a \otimes \omega^b$$

$$\mathit{TypeVIII} \quad \begin{aligned} \omega^1 &= -\sinh\psi\sinh\theta d\phi \ + \ \cosh\psi d\theta \\ \omega^2 &= -\cosh\psi\sinh\theta d\phi \ + \ \sinh\psi d\theta \\ \omega^3 &= \cosh\theta d\phi \ + \ d\psi \end{aligned}$$

$$\mathit{TypeIX} \quad \begin{aligned} \omega^1 &= \sin\psi\sin\theta d\phi \ + \ \cos\psi d\theta \\ \omega^2 &= -\cos\psi\sin\theta d\phi \ + \ \sin\psi d\theta \\ \omega^3 &= \cos\theta d\phi \ + \ d\psi \end{aligned}$$

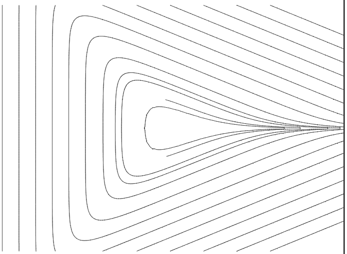
Hamiltonian Formulation

$$\delta S = \delta \int \Big(p_\alpha \alpha' + p_+ {\beta_+}' + p_- {\beta_-}' - NH\Big) dt = 0$$

$$H = \frac{e^{-3\alpha}}{24\pi}\Big(-p_\alpha^2 + p_+^2 + p_-^2 + \mathcal{V}\Big) \qquad \mathcal{V} = -12\pi^2 e^{4\alpha} U^{(B)}(\beta_+, \beta_-)$$

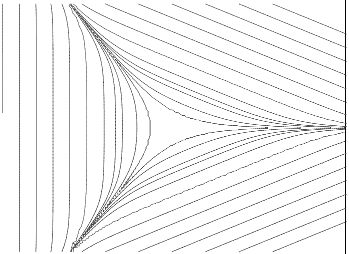
Mixmaster equipotential lines in the $\beta_\pm$-plane

Bianchi type VIII



$$U^{\mathrm{VIII}} = e^{-8\beta_+} + 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}\left(\cosh(4\sqrt{3}\beta_-) - 1\right)$$

Bianchi type IX



$$U^{\mathrm{IX}} = e^{-8\beta_+} - 4e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-) + 2e^{4\beta_+}\left(\cosh(4\sqrt{3}\beta_-) - 1\right)$$

Same behaviour for large values of β when $|\beta_-| \ll 1$

$\beta_+ \longrightarrow +\infty : \quad U(\beta) \simeq 48{\beta_-}^2 e^{4\beta_+}$

$\beta_+ \longrightarrow -\infty : \quad U(\beta) \simeq e^{-8\beta_+}$

Mixmaster dynamics

- The Universe evolution is described as **the motion of a point-like particle in the potential well.**
- The asymptotic region where the potential terms are significant is $$\beta_{wall} = \frac{\alpha}{2} - \frac{1}{8}\ln(3H^2)$$
- $H = \mathrm{const.}$ inside the potential well $\Rightarrow$ $|\beta'_{wall}| = 1/2$

$\boxed{\beta\text{-point moves twice as fast as the receding potential wall}}$

- **the particle will collide against the wall and will be reflected from one straight-line motion (Bianchi I) to another one.**

Reflection Law

$$\boxed{\sin\theta_f - \sin\theta_i = \frac{1}{2}\sin(\theta_i + \theta_f)}$$

θ_i is the angles of incidence and θ_f is the angle of reflection
This is the analogues of $u_f = u_i - 1$.

"Stationary walls": Misner-Chitré variables

[Chitré *PhD Thesis* (1972)]

One set of this kind of variables is:
$1 \leq \xi < \infty$, $0 \leq \theta < 2\pi$
and τ plays the role of a "radial" coordinate

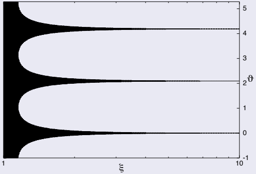
$$\begin{cases} \alpha &= -e^{\tau}\xi \\ \beta_+ &= e^{\tau}\sqrt{\xi^2-1}\cos\theta \\ \beta_- &= e^{\tau}\sqrt{\xi^2-1}\sin\theta \end{cases}$$

Anisotropy parameters and the asymptotic domain

$$Q_1 = \frac{1}{3} - \frac{\sqrt{\xi^2-1}}{3\xi}\left(\cos\theta + \sqrt{3}\sin\theta\right)$$

$$Q_2 = \frac{1}{3} - \frac{\sqrt{\xi^2-1}}{3\xi}\left(\cos\theta - \sqrt{3}\sin\theta\right)$$

$$Q_3 = \frac{1}{3} + 2\frac{\sqrt{\xi^2-1}}{3\xi}\cos\theta\,.$$



$\boxed{\text{the potential walls are static with respect to the time variable}}$

Variational principle

$$S = \int\left(p_\xi\xi' + p_\theta\theta' + p_\tau\tau' - \frac{Ne^{-2\tau}}{24D}H\right)dt$$

Hamiltonian

$$H = -{p_\tau}^2 + {p_\xi}^2\left(\xi^2-1\right) + \frac{{p_\theta}^2}{\xi^2-1} + 24Ve^{2\tau}$$

$D = \exp\{-3\xi e^{\tau}\}$ is the determinant of the 3-metric.

Cosmological Chaos

The Gauss Map

[BKL and JD Barrow *Phys Rev Lett* **46**, 963 (1981)]

Continuous fraction

- Every s-th era is described by: $u^{(s)}_{max},\ u^{(s)}_{max}-1,\ u^{(s)}_{max}-2,\dots,u^{(s)}_{min}$.

u-decomposition: $u^{(s)} = k^{(s)} + x^{(s)}$
initial values: $k^{(0)} + x^{(0)}$

$$\boxed{x^{(0)} = \cfrac{1}{k^{(1)} + \cfrac{1}{k^{(2)} + \cfrac{1}{k^{(3)} + \dots}}}}$$

1. a rational number would have a finite expansion
2. periodic expansion represents quadratic irrational numbers
3. irrational numbers have infinite expansion.

Statistic distribution of the eras

- k and x are not independent $\Rightarrow$ they admit a stationary probability distribution

$$w(k,x) = \frac{1}{(k+x)(k+x+1)\ln 2} \qquad \boxed{w(u) = \frac{1}{u(u+1)\ln 2}} \qquad (u = k + x)$$

- The existence of the Gauss map was firstly demonstrated by BKL

PROPERTIES

1. positive metric- and topologic-entropy
2. *weak Bernoulli properties* (cannot be finitely approximated)
3. *strongly mixing* and *ergodic*

The Invariant Measure

Discrete dynamics

[Chernoff and Barrow *Phys Rev Lett* **50**, 134 (1983)]

$$\alpha = 0 \qquad \frac{\partial \alpha}{\partial \tau} = s p_2(u) = \frac{s(1+u)}{1+u+u^2}$$

$$\beta = \frac{\Sigma x}{(x+1+u)} \qquad \frac{\partial \beta}{\partial \tau} = s p_1(u) = -\frac{su}{(1+u+u^2)}$$

$$\gamma = \frac{\Sigma(1+u)}{x+1+u} \qquad \frac{\partial \gamma}{\partial \tau} = s p_3(u) = \frac{su}{(1+u+u^2)}$$

$$(u', x') = \begin{cases} (u-1, x/(1+x)) & \text{if } \infty > u > 1 \\ (1/u - 1, 1 + 1/x) & \text{if } 1 > u > 0 \end{cases}$$

- This map is a generalization of the Baker transformation
- **Properties**:
  dense periodic orbits, no integral of the motion, ergodic, strongly mixing.
- Associated invariant measure:

$$\boxed{\mu(u, x) = \frac{1}{\ln(2)(1+ux^2)}}$$

Continuous Dynamics

[Kirillov and Montani *Phys Rev D* **56**, 6225 (1997)]

- In Misner-Chitré representation $\Rightarrow$ existence an asymptotic energy like constant
- Statistical-mechanics point of view:
  The dynamics can be described as a microcanonical ensemble
- Liouville invariant measure: $\boxed{d\mu = d\xi d\theta d\phi \frac{1}{8\pi^2}}$

The invariant measure provides the complete equivalence between the BKL piece-wise description and the Misner-Chitre continuous one (Artins theorem).

Lyapunov exponent for Mixmaster

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Lyapunov exponent for Mixmaster

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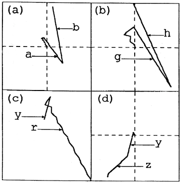
Dependence of the Lyapunov exponent on the choice of the time variable

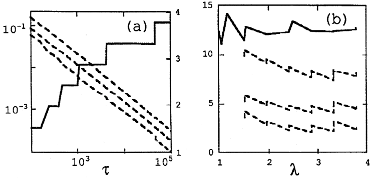
B. K. Berger *Gen Rel Grav* **7**, 203 (1990)

Through numerical simulations, the Lyapunov exponents were evaluated along some trajectories in the $(\beta_+,\ \beta_-)$ plane for different choices of the time variable:

1. τ (BKL)
2. Ω (Misner)
3. λ, the "mini-superspace" one, $d\lambda = |-p_\Omega^2 + p_+^2 + p_-^2|^{1/2} d\tau$

The same trajectory giving zero Lyapunov exponent for τ or Ω-time, fails for λ.





Left panel:
trajectories in the anisotropy plane $(\beta_+,\ 0,\ \beta_-)$.

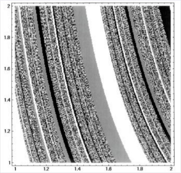
Center and right panel:
solid line indicates λ vs. τ. dashed line indicates the three positive Lyapunov exponents

Strange Repellors

[Cornish and Levin *Phys.Rev.D* **55**, 7489 (1997) and *Phys.Rev.Lett.***78**, 998 (1997)]

Boundary Basin Method

- By exploiting techniques originally developed to study chaotic scattering, they found a fractal structure, $\boxed{\text{THE STRANGE REPELLOR}}$.



- It is the collection of all Universes periodic in $(u,\ v)$
- An aperiodic one will experience a transient age of chaos if it brushes against the repellor.
- The fractal pattern was exposed in both the exact and in the discrete Einstein equations.
- It is independent of the time coordinate and the chaos reflected in the fractal weave is unambiguous.

Three essential fallacies:

1. the case-points proceeds never reaching the singularity.
2. the "most frequent" dynamical evolution is the one in which the point enters the corner with the velocity *not parallelly* oriented towards the corner's bisecting line
3. the artificial opening up of the potential corners could be creating the fractal nature of it.

Covariant Lyapunov exponent

[Imponente and Montani *Phys.Rev.D* **63**, 103501 (2001)]

Adopting Misner-Chitré like variables, it is possible to demonstrate that
Lyapunov exponent is invariant under time reparametrizations

Generic-time formulation

Introduce a generic function of the time variable

$$H = -\frac{{p_\tau}^2}{(d\Gamma/d\tau)^2} + {p_\xi}^2\left(\xi^2-1\right) + \frac{{p_\theta}^2}{\xi^2-1} + 24Ve^{2\Gamma}$$

Reduce the dynamics with the standard ADM technique

$$-p_\tau \equiv \frac{d\Gamma}{d\tau}\mathcal{H}_{ADM} = \frac{d\Gamma}{d\tau}\sqrt{\varepsilon^2 + 24Ve^{2\Gamma}}$$

$$\varepsilon^2 \equiv \left(\xi^2-1\right){p_\xi}^2 + \frac{{p_\theta}^2}{\xi^2-1}$$

Asymptotic dynamics and energy constant of motion

ADM Hamiltonian becomes (asymptotically) an integral of motion

$$\forall\{\xi,\theta\}\in\Pi_Q \quad \begin{cases} \dfrac{\partial\mathcal{H}_{ADM}}{\partial\Gamma} = 0 = \dfrac{\partial E}{\partial\Gamma} \\ \mathcal{H}_{ADM} = \sqrt{\varepsilon^2 + 24\,U} \cong \varepsilon = E = \text{const.} \end{cases}$$

Covariant Lyapunov exponent

[Imponente and Montani *Phys.Rev.D* **63**, 103501 (2001)]

Billiard dynamics

The standard Jacobi method applied to Mixmaster yields the metric

$$ds^2 = E^2\left[\frac{d\xi^2}{\xi^2-1} + \left(\xi^2-1\right)d\theta^2\right]$$

The above metric describes a constant negative curvature surface

$$R = -2/E^2$$

Covariant Lyapunov exponent

introduce orthonormal tetradic basis:

- v^i is the geodesic field
- w^i is parallely transported.

$$\begin{cases} v^i &= \left(\frac{1}{E}\sqrt{\xi^2-1}\cos\phi,\ \frac{1}{E}\frac{\sin\phi}{\sqrt{\xi^2-1}}\right) \\ w^i &= \left(-\frac{1}{E}\sqrt{\xi^2-1}\sin\phi,\ \frac{1}{E}\frac{\cos\phi}{\sqrt{\xi^2-1}}\right) \end{cases}$$

Project the geodesic deviation equation along the vector w^i the component Z satisfies

$$\frac{d^2Z}{ds^2} = \frac{Z}{E^2} \qquad Z(s) = c_1 e^{\frac{s}{E}} + c_2 e^{-\frac{s}{E}}$$

This scalar expression is completely independent of the choice of the variables.

Invariant Lyapunov exponent $$\lambda_v = \sup \lim_{s\to\infty} \frac{\ln\left(Z^2 + \left(\frac{dZ}{ds}\right)^2\right)}{2s} = \frac{1}{E} > 0$$

Chaos Covariance in the Generic Solution [R.Benini and G.Montani *Phys Rev D* **70**, 103527 (2004)]

Hamiltonian Formulation of the generic problem

Three-metric tensor describing a generic model

$$h_{\alpha\beta} = e^{q_a}\delta_{ad}O^a_b O^d_c \partial_\alpha y^b \partial_\beta y^c$$

Hamiltonian Formulation

$$S = \int_{\Sigma\times R} dt d^3x \left(p_a \partial_t q^a + \Pi_d \partial_t y^d - NH - N^\alpha H_\alpha\right)$$

$$H = \frac{1}{\sqrt{h}}\left[\sum_a (p_a)^2 - \frac{1}{2}\left(\sum_b p_b\right)^2 - h\,^{(3)}R\right]$$

$$H_\alpha = \Pi_a \partial_\alpha y^a + p_a \partial_\alpha q^a + 2p_a (O^{-1})^b_a \partial_\alpha O^a_b$$

Gauge conditions

$$\partial_t y^d = N^\alpha \partial_\alpha y^d$$

$$N = \frac{\sqrt{h}}{\sum_a p_a}\left(N^\alpha \partial_\alpha \sum_b q^b - \partial_t \sum_b q^b\right)$$

Chaos Covariance in the Generic Solution [R.Benini and G.Montani *Phys Rev D* **70**, 103527 (2004)]

The super-momentum constraint can be explictly solved

$$\Pi_b = -p_a \frac{\partial q^a}{\partial y^b} - 2p_a (O^{-1})^c_a \frac{\partial O^a_c}{\partial y^b}$$

Structure of the Ricci scalar

$$U = \frac{D}{|J|^2}\,{}^{(3)}R = \sum_a \lambda_a^2 D^{2Q_a} + \sum_{b\neq c} D^{Q_b+Q_c}\mathcal{O}\left(\partial q, (\partial q)^2, y, \eta\right)$$

$$\lambda_a^2 \equiv \sum_{k,j}\left(O^a_b \vec{\nabla} O^a_c \left(\vec{\nabla} y^c \wedge \vec{\nabla} y^b\right)^2\right)$$

It can be demonstrated that

$$\boxed{\to U = \sum_a \Theta(Q_a)}$$

in the limit toward the Singularity

The same analysis developed for the homogeneous case

Homogeneous *n*-dimensional generalizations

[P. Halpern *Gen Rel Grav* **35**, 251 (2003)]

- Halpern studied the 14 possible 4-d Lie algebra (named $G0 - G14$).
- Mixmaster generalization is recognized in $G13$ because it's the only one with all structure constants equal to 1

5-d Homogeneous Einstein equations

$$2\alpha_{\tau\tau} = \left[(b^2 - c^2)^2 - a^4\right] d^2$$
$$2\beta_{\tau\tau} = \left[(a^2 - c^2)^2 - b^4\right] d^2$$
$$2\gamma_{\tau\tau} = \left[(b^2 - a^2)^2 - c^4\right] d^2$$

$$\delta_{\tau\tau} = 0$$
$$\alpha_{\tau\tau} + \beta_{\tau\tau} + \gamma_{\tau\tau} + \delta_{\tau\tau} = + 2\alpha_\tau\gamma_\tau + 2\alpha_\tau\delta_\tau + 2\beta_\tau\gamma_\tau + + 2\beta_\tau\delta_\tau + 2\gamma_\tau\delta_\tau$$

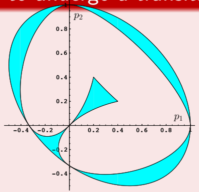
- From which Kasner relations follow and so on...

What is the difference? the conditions needed to undergo a transition

$$1 - 3p_1^2 - 3p_2^2 - 2p_1p_2 + 2p_1 + 2p_2 \geq 0$$

and one of the following ones

$$3p_1^2 + p_2^2 + p_1 - p_2 - p_1p_2 < 0$$
$$3p_2^2 + p_1^2 + p_2 - p_1 - p_1p_2 < 0$$
$$3p_1^2 + p_2^2 - 5p_1 - 5p_2 + 5p_1p_2 + 2 < 0$$



A region where the p_i's are all greater than zero exists!!

HOMOGENEOUS *n*-D MODELS ARE CHAOTIC ONLY IN 3 SPACE DIMENSIONS

Inhomogeneous n-d models

[J. Demaret et al *Phys Lett B* **175**, 129 (1986)]

n- dimensional Einstein dynamics

- $(d+1)$ Einstein equations $\quad {}^{(d+1)}R_{ik} = 0$
- The metric admits a generalized Kasner solution: $\quad h_{\alpha\beta} = t^{2p_a} l^a_\alpha l^a_\beta$
- Generalized Kasner relations $\quad \sum p_a(x^\gamma) = \sum p_a^2(x^\gamma) = 1$
- In each point, the conditions fix one point on a $(d-2)$-dimensional sphere.

Stability of the solution

- Each single step of the solution is stable if $\quad \lim_{t\to 0} t^2\ {}^{(d)}R^b_a = 0$
- The only terms capable to perturb the Kasner behavior have the form $t^{2\alpha_{abc}}$

$$\alpha_{abc} = 2p_a + \sum_{d\neq a,b,c} p_d\ , \qquad (a\neq b, a\neq c, b\neq d)$$

Two possibilities:

1. The Kasner exponents can be chosen in an open region of the Kasner sphere defined in so as to make α_{abc} positive for all triples a, b, c
2. The conditions $\alpha_{abc}(x^\gamma) > 0 \quad \forall(x^1, \ldots, x^d)$ are in contradiction, and one must impose extra conditions on the functions l and their derivatives.

Inhomogeneous n-d models

[J. Demaret et al *Phys Lett B* **175**, 129 (1986)]

The answer? It depends on the *number of dimensions!*

It can be shown that,

- for $3 \leq d \leq 9$:
  at least the smallest α, i.e. $\alpha_{1,d-1,d}$ results to be always negative
  and the evolution of the system to the singularity consists of an infinite number of Kasner epochs
- for $d \geq 10$:
  an open region exists of the $(d-2)$-dimensional Kasner sphere where $\alpha_{1,d-1,d}$ takes positive values, the so-called **Kasner Stability Region (KSR).**
  and the existence of the KSR, implies a final stable Kasner-like regime.

Matter-Filled spaces

If we add a scalar field ϕ to the dynamics

$$2H = -p_\Omega^2 + p_+^2 + p_-^2 + p_\phi^2 + V + e^{6\Omega}\mathcal{V}(\phi)$$

with a BKL-like analysis we obtain a metric

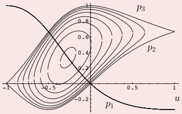
$$ds^2 = dt^2 - \sum_{i=1}^{3} t^{2p_i}(dx^i)^2$$

$$p_1 + p_2 + p_3 = 1\,, \quad p_1^2 + p_2^2 + p_3^2 = 1 - q^2$$

$$\phi(t) = q\ln(t) + \phi_0.$$

OSCILLATORY REGIME SUPPRESSED!

The new Kasner relations allows the existence of a region where the p_i's are all greater than zero



and a solution with $p_i \geq 0$ is a Kasner epoch stable toward the singularity

[Berger *Phys Rev D* **61**, 023508 (1999)]

Hamiltonian

$$\Pi^\alpha_\beta \Pi^\beta_\alpha - \frac{\Pi^2}{n-1} + \frac{1}{2}\Pi^\alpha \Pi_\beta + h\left(\frac{F_{\alpha\beta}F^{\alpha\beta}}{4} - R\right) = 0$$

A BKL-like analysis yield a BKL map

$$p_1' = \frac{-p_1}{1+\frac{2}{n-2}p_1}\,, \quad p_a' = \frac{p_a + \frac{2}{n-2}p_1}{1+\frac{2}{n-2}p_1}$$

$$\widetilde{\lambda}_1' = \widetilde{\lambda}_1\,, \quad \frac{\widetilde{\lambda}_a'}{\widetilde{\lambda}_a} = \left(1 - 2\frac{(n-1)\,p_1}{(n-2)\,p_a + np_1}\right)$$

$$\vec{\ell}_a' = \vec{\ell}_a + \sigma_a \vec{\ell}_1, \quad \sigma_a = -2\frac{(n-1)\,p_1}{(n-2)\,p_a + np_1}\frac{\widetilde{\lambda}_a}{\widetilde{\lambda}_1}$$

Vector field restores chaos in homogeneous model in any number of dimensions!

[R.Benini, A. Kirillov, G.Montani *Class Quantum Grav* **22**, 1483 (2005)]

Non Classical Properties

WDW Equation in Misner variables

Adopting the canonical representation, address the WDW eqn

$$\hat{H}\psi = e^{-3\alpha}\left[-\frac{\delta^2}{\delta\alpha^2} + \frac{\delta^2}{\delta\beta_+^2} + \frac{\delta^2}{\delta\beta_-^2}\right]\psi - e^{\alpha}V\psi = 0$$

Misner proposal:

[C. W. Misner *Phys Rev Lett* **22**, 1071 (1969)]

Approximate energy levels in the triangular box with those of the standard box

$$E_n(\alpha) = \pi\left(\frac{4}{3^{3/2}}\right)^{1/2}\frac{|n|}{\alpha},$$

α- dependent eigenfunctions

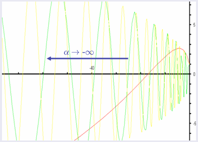
[Imponente and Montani *IJMP D* **12**, 977 (2003)]

Ansatz: general solution of the form $\psi = \sum_n \Gamma_n(\alpha)\varphi_n(\alpha, \beta_+, \beta_-)$

Adabatic approximation: neglect $\partial_\alpha \phi_n$

$$\Gamma_n(\alpha) = C_1\sqrt{\alpha}\sin\left(\frac{1}{2}\sqrt{p_n}\ln\alpha\right) + C_2\sqrt{\alpha}\cos\left(\frac{1}{2}\sqrt{p_n}\ln\alpha\right)$$

$$\sqrt{p_n} = \sqrt{k_n^2 - 1} \qquad k_n^2 = \left(\frac{2\pi}{3}\right)^{3/2}|n|^2 .$$



No quantum Universe: mean occupation number n is constant toward the singularity.

If the initial state is classical, backwards it maintains a semiclassical character.

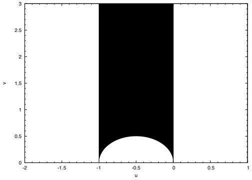
The Poincaré plane

Poincaré upper half-plane representation

$$\delta S_{\Pi_Q} = \delta \int d\tau (p_u \dot{u} + p_v \dot{v} - H_{ADM}) = 0 \qquad H_{ADM} = v\sqrt{p_u^2 + p_v^2}$$

The dynamics is restriced in a portion Π_Q of the Lobatchevsky plane

$$Q_1(u,v) = -u/d \geq 0$$
$$Q_2(u,v) = (1+u)/d \geq 0 \qquad d = 1 + u + u^2 + v^2$$
$$Q_3(u,v) = (u(u+1) + v^2)/d \geq 0$$



The billiard has a finite measure, and has an open region at infinity together with two points on the absolute $(0,0)$ and $(-1,0)$.

$$d\mu = \frac{1}{\pi}\frac{du\,dv}{v^2}\frac{d\phi}{2\pi}$$

Schroedinger Dynamics

If we assume a generic order a

$$i\frac{\partial \Phi}{\partial \tau} = \hat{H}_{ADM}\Phi = \sqrt{-v^2\frac{\partial^2}{\partial u^2} - v^{2-a}\frac{\partial}{\partial v}\left(v^a\frac{\partial}{\partial v}\right)}\;\Phi$$

WKB analysis - vs - Liouville theorem [R. Benini and G. Montani *Class Quantum Grav* **24**, 387 (2007)]

The quantum dynamics is non local!

We assume that operators $\hat{H}$ and $\hat{H}^2$ have same spectrum and eigenfunctions

Looking for a WKB solution of the form

$$\Psi(u, v, E) = \sqrt{r(u, v, E)}e^{\imath\sigma(u,v,E)/\hbar}$$

we obtain for r the equation

$$C\partial_u r + \sqrt{\frac{E^2}{v^2} - C^2}\partial_v r + \frac{a(E^2 - C^2v^2) - E^2}{v^2\sqrt{E^2 - C^2v^2}} r = 0$$

The Mixmaster admits an energy like constant of motion

We can analyze it as a microcanonical ensemble studying the continuity equation

The distribution function $\tilde{w}$ satisfies the following eqn in the configuration space

$$\frac{\partial\tilde{w}}{\partial u} + \sqrt{\left(\frac{E}{Cv}\right)^2 - 1}\,\frac{\partial\tilde{w}}{\partial v} + \frac{E^2 - 2C^2v^2}{Cv^2\sqrt{E^2 - (Cv)^2}}\tilde{w} = 0$$

r is the distribution function in the WKB limit

$\tilde{w}$ is the distribution function in classical real

$\Rightarrow$

Requiring that they coincide we get a unique value for a

$$a = 2 \Rightarrow \boxed{\hat{v}^2\hat{p}_v^2 \longrightarrow -\hbar^2\frac{\partial}{\partial v}\left(v^2\frac{\partial}{\partial v}\right)}$$

The Spectrum of the Mixmaster

[R. Benini and G. Montani *Class Quantum Grav* **24**, 387 (2007)]

Eingenvalue equation (with $a = 2$)

$$\left[v^2 \frac{\partial^2}{\partial u^2} + v^2 \frac{\partial^2}{\partial v^2} + 2v \frac{\partial}{\partial v} + \left(\frac{E}{\hbar}\right)^2\right] \Psi(u, v, E) = 0$$

$\Rightarrow$

If $\Psi(u, v, E) = \psi(u, v, E)/v \Rightarrow$ eigenvalue problem for the Laplace-Beltrami operator in H

$$\nabla_{LB}\psi \equiv v^2 \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right) \psi = E_s \psi$$

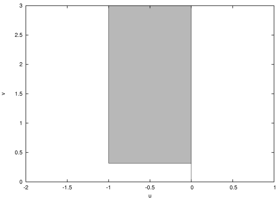
General solution (no boundary conditions): Eigenfunctions and Spectrum

$$\Psi(u, v, E) = a v^{s-1} + b v^{-s} + \sum_{n \neq 0} a_n \frac{K_{s-1/2}(2\pi|n|v)}{\sqrt{v}} e^{2\pi i n u} ; \qquad E^2 = s(1-s)$$

Energy spectrum

Dirichlet boundary conditions:

$$\boxed{\Psi(\partial\Gamma_Q) = 0}$$



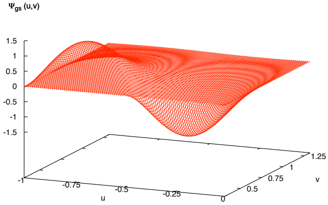
The Spectrum of the Mixmaster

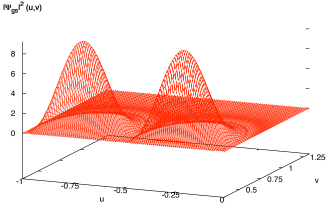
[R. Benini and G. Montani *Class Quantum Grav* **24**, 387 (2007)]

Spectrum $E^2/\hbar^2 = t^2 + 1/4$

$s = 1/2 + it$
with t such that $K_{it}(2n) = 0, \quad n \in \mathbb{N}$

$E^2 = t^2 + \frac{1}{4}$
19.831
40.357
49.474
63.405
87.729
89.250
116.329
128.234
138.739
146.080





Conclusions and Bibliography

Conclusions

Final Considerations

- The astonishing result consists of the possibility to extend the oscillatory regime to the generic inhomogeneous case, as far as a sub-horizon geometry is concerned.
- This very general dynamics does not provide only the important, but somehow academical, proof about the existence of the singularity but it also represents the real physical arena to implement any reliable theory of the Universe birth.
- Indeed, both from a classical and from a quantum point of view, the inhomogeneous Mixmaster offers a scenario of full generality to investigate the viability of a theoretical conjecture

Open Issues

- An important issue will be to fix the chaotic features as expectable properties of the Universe origin, when a convincing proposal for the quantization of gravity will acquire the proper characteristics of a Theory.
- The transition to the classical limit of an expanding Universe.

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