Quantum Isotropization Mechanism for the Mixmaster Model

Motivation:
Analyze the evolution of the wave-function of the Universe starting from anisotropic conditions. The treatment point at its probabilistic interpretation through a Born-Hoppeneimer / WKB approximation.

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Summary of talk

I. Wheeler-DeWitt and WKB probabilistic approaches: interpreting the wave-function of the Universe

II. Homogeneus Bianchi Spaces and their temporal coordinate

III. Bianchi IX Space and small-anisotropies limit of the model

IV. Wave-function for the quasi-isotropic model and its behaviour

V. Classical limit of the model

I. Wheeler-DeWitt and WKB probabilistic approaches

- In canonical quantum cosmology, the wave-function $\Psi\left[h_{ij}(x),\Phi(x)\right]$ is defined on an infinite-dimensional space of all possible 3-geometries and matter field configurations, known as **superspace**: to make the problem tractable, all but a finite number of degrees of freedom must be "frozen out" $\Rightarrow$ the resulting finite-dimensional superspace is known as **mini-superspace**.

- Action for the minisuperspace homogeneus model:

$$S = \int dt\left\{p_\alpha \dot{h^\alpha} - N[g^{\alpha\beta}p_\alpha p_\beta + U(h)]\right\} \tag{1}$$

where

$$U = h^{1/2}[\tilde{W}(\Phi) -^{(3)} R] \tag{2}$$

with $h = |deth_{ij}|$, $\tilde{W}(\Phi)$ is the potential energy of matter fields, and ${}^{(3)}R$ is the curvature of three-space with metric h_{ij}.

- Wheeler-DeWitt equation for the action (1):

$$\left(\nabla^2 - U\right)\Psi = 0 \tag{3}$$

i.e. an n-dimensional Klein-Gordon equation $\Rightarrow$ probability dP to find the system in a certain configuration-space element $d\Omega$ can be negative $\Rightarrow$ difficulty to give a physical meaning to the wave-function

- Born-Hoppeneimer/WKB approximation: the superspace variables are divided into two classes: **semiclassical** (indicated by h^α, $\alpha = 1, ..., n-m$) and **quantum** (indicated by q^ν, $\nu = 1, ..., m$). We assume that the quantistic variables don't affect the semiclassical ones $\Rightarrow$ "the variables q^ν correspond to a small subsystem of the

Universe"; the smallness of the subsystem is mathematically manifested by a small parameter λ proportional to $\hbar$. By this way, the Wheeler-DeWitt equation (3) and the wave-function of the Universe can be written, respectively, as:

$$\left(\nabla_0^2 - U_0 - H_q\right)\Psi = 0 \tag{4}$$

$$\Psi(h,q) = A(h)e^{iS(h)}\chi(h,q) \tag{5}$$

In the lowest order in λ, we find the **Hamilton-Jacobi** equation for S:

$$g^{\alpha\beta}\left(\nabla_\alpha S\right)\left(\nabla_\beta S\right) + U = 0 \tag{6}$$

in the next order the equation for the amplitude A :

$$2\nabla A\nabla S + A\nabla^2 S = 0 \tag{7}$$

and, finally, the equation for the subsystem χ:

$$i\frac{\partial\chi}{\partial t} = N(t)H_q\chi \tag{8}$$

where the terms of higher order in λ have been neglected. The splitting into two class of the variables leads to introduce two corresponding conserved currents and the probability distribution becomes:

$$\rho(h,q,t) = \rho_0(h,t)\,|\chi(q,h(t),t)|^2 \qquad (9)$$

The probability

$$dP = jd\Sigma \qquad (10)$$

where $d\Sigma$ is the surface element on the equal-time surfaces, is positive semidefinte, with the two subprobability normalizable to the unity.

II. Homogeneus Bianchi Spaces and their temporal coordinate

Because of quantum fluctuations, a quantum Universe has to be described by a generic inhomogeneous model; the super-Hamiltonian in (1), expressed by the Misner variables $(\alpha, \beta_+, \beta_-)$, is:

$$\mathcal{H} = \kappa \left[-\frac{p_a^2}{a} + \frac{\left(p_+^2 + p_-^2\right)}{a^3} \right] + \frac{a}{4\kappa} V(\beta_+, \beta_-) + U(a) \tag{11}$$

where

$$U(a) = -\frac{a}{4\kappa} + \frac{\Lambda}{\kappa} a^3 \qquad . \tag{12}$$

and the different Bianchi Spaces are contained in the term $V(\beta_+, \beta_-)$. a is the scale-factor, describing the volume of the Universe, β_+, β_- are two anisotropy parameters. We have choose to assign to the scale factor a the role of semi-classical variables, moreover, looking at the signature in (11), is clear its role as time-coordinate: following the approximation previously exposed, and scaling the lapse-function and the scale factor a, we obtain:

$$i\frac{\partial\chi}{\partial\tau} = H_\tau\chi \tag{13}$$

where

$$H_\tau = \left[-\Delta_\beta + \omega^2(\tau)V\right] \tag{14}$$

$$\omega^2(\tau) = \frac{c}{\tau^{4/3}} \tag{15}$$

and

$$a = \left(\frac{\kappa}{12\tau\sqrt{\Lambda}}\right)^{\frac{1}{3}} \tag{16}$$

valid in the limit

$$a \gg \frac{1}{\sqrt{\Lambda}} \tag{17}$$

III. Bianchi IX Space and small-anisotropies limit of the model

We have used the most general homogeneus space, the Bianchi IX one, that, in the limit of small-anisotropies, is represented by

$$V(\beta_+, \beta_-) \approx 8(\beta_+^2 + \beta_-^2) \tag{18}$$

obtaining:

$$H_\tau = \left[-\Delta_\beta + \Omega^2(\tau)(\beta_+^2 + \beta_-^2)\right] \tag{19}$$

and

$$\Omega^2(\tau) = \frac{C}{\tau^{4/3}} \tag{20}$$

The Hamiltonian (19) represents a bi-dimensional time-dependent armonic oscillator mechanical system.

IV. Wave-function for the quasi-isotropic model and its behaviour

Through the introduction of a new kind of operator, the **exact generalized invariant**, whose eigenstates are connected with those of the armonic oscillator time-dependent hamiltonian, and through the introduction of creation and distruction operators, we find

$$\chi(\beta_\pm,\tau) = \chi_+(\beta_+,\tau)\chi_-(\beta_-,\tau) \tag{21}$$

with:

$$\chi_\pm(\beta_\pm,\tau) = e^{i\alpha_{n_\pm}(\tau)}\left(\frac{1}{n_\pm!2^{n_\pm}\rho}\right)^{\frac{1}{2}} H_n\left(\frac{\beta_\pm}{\rho}\right)\exp\left[\frac{i}{2}\left(\frac{\rho'}{\rho}+\frac{i}{\rho^2}\right)\beta_\pm^2\right] \tag{22}$$

with ρ any solution of a second order differential equation connected to the exact invariant ,n a new **quantum number** due to the introduction of the creation and distruction operators and H_n the Hermite polynomials. From (22), we obtain:

$$|\chi|^2 \propto \frac{1}{\rho^2}\left|H_{n_+}\left(\frac{\beta_+}{\rho}\right)\right|^2\left|H_{n_-}\left(\frac{\beta_-}{\rho}\right)\right|^2 e^{-2\frac{\beta_+^2}{\rho^2}} e^{-2\frac{\beta_-^2}{\rho^2}} \tag{23}$$

so that the probability density go to nullify exponentially once the anisotropy variable $\beta^2 = \beta_+^2 + \beta_-^2$ goes larger:

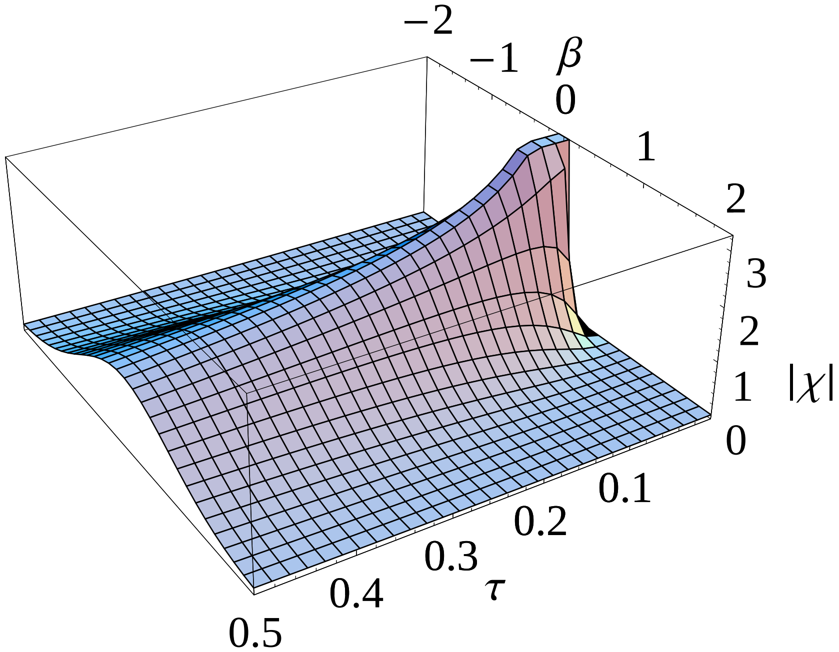
$$|\chi|^2 \propto \frac{1}{\rho^2}\left|H_{n_+}\left(\frac{\beta_+}{\rho}\right)\right|^2\left|H_{n_-}\left(\frac{\beta_-}{\rho}\right)\right|^2 e^{-\frac{\beta^2}{\rho^2}} \overset{\beta\to\infty}{\longrightarrow} 0 \tag{24}$$

Moreover, considering the behaviour of the ρ function, moving away from initial singularity:

$$\rho(\tau) \propto \tau^{1/2}\left(1 + \frac{\tau^{-\frac{2}{3}}}{9C}\right)^{\frac{1}{2}} \overset{\tau\to 0}{\longrightarrow} \tau^{1/6} \tag{25}$$

we obtain:

$$|\chi(\beta,\tau)|^2 \propto \frac{1}{\tau^{1/6}} e^{\frac{\beta^2}{\tau^{1/3}}} \left| H_{n_+}\left(\frac{\beta_+}{\tau^{1/6}}\right)\right|^2 \left| H_{n_-}\left(\frac{\beta_-}{\tau^{1/6}}\right)\right|^2 \overset{\tau\to 0}{\longrightarrow} \frac{1}{\tau^{1/6}}\delta(\beta,0) \tag{26}$$



Wave-function shape on its fundamental state $n = 0$.

As it's easy to be seen, **the probability density is not null only in the region in which the anisotropies go down**

V. Classical limit of the model

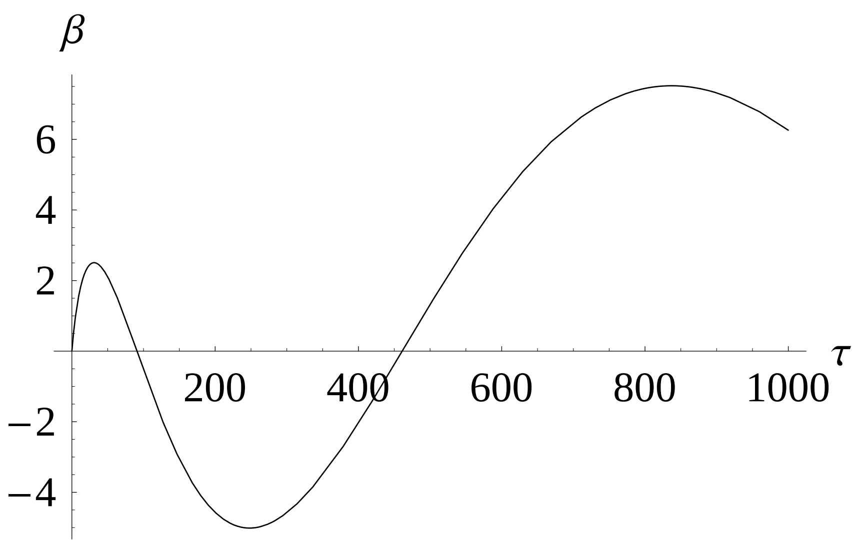
Splitting the S-function into two terms

$$S(a, \beta_+, \beta_-) = S_0(a) + S_1(\beta_+, \beta_-) \tag{27}$$

we find the same results; i.e., in the limit $a \gg 1/\sqrt{\Lambda}$, we obtain the equation for $\beta_\pm$

$$\beta_\pm = D_\pm \tau^{1/2} J_{3/2}\left(3\sqrt{c}\tau^{1/3}\right) \tag{28}$$

where $J_{3/2}$ is the Bessel function of the first kind.



Classical behaviour of anisotropies variables on respect to the time τ; $D_\pm = 1$