Extended Approach to the Canonical Quantization in the Minisuperspace

Motivations:
Find a unique framework which phenomenologically describes the effective Friedmann dynamics of LQC and braneworlds scenario. Deformed minisuperspace (Heisenberg) algebra which is related to the κ-Pincaré one. This way, a non-trivial link between these theories is founded from a low energy perspective.

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Summary of talk

I. Deformed quantum mechanics

II. Deformed FRW dynamics

III. Deformed uncertainty principle

IV. Discussion and conclusions

I. Deformed quantum mechanics

Deformed Heisenberg algebra such that:
(i) No deforms rotation and translation groups
(ii) Ordinary Lie one is recovered in a limit

$$
\begin{aligned}
[\mathbf{q}_i, \mathbf{p}_j] &= i\delta_{ij}\sqrt{1 \pm \alpha \mathbf{p}^2} \quad \mathbf{p}^2 = \mathbf{p}_i \mathbf{p}^i \\
[\mathbf{p}_i, \mathbf{p}_j] &= 0 \\
[\mathbf{q}_i, \mathbf{q}_j] &= \mp i\alpha \mathbf{J}^{\alpha}_{ij}
\end{aligned}
\tag{1}
$$

Generators of rotation group

$$
\mathbf{J}^{\alpha}_{ij} = \frac{1}{\sqrt{1 \pm \alpha \mathbf{p}^2}} \Big(\mathbf{q}_i \mathbf{p}_j - \mathbf{q}_j \mathbf{p}_i \Big) \tag{2}
$$

In 3-dim:
$\mathbf{J} \in SU(2)$, $[\mathbf{J}^{\alpha}_i, \mathbf{q}_j] = i\epsilon_{ijk}\mathbf{q}_k$, $[\mathbf{J}^{\alpha}_i, \mathbf{p}_j] = i\epsilon_{ijk}\mathbf{p}_k$

No sign in (1) is selected at all by the assumptions

Algebra (1) can be obtained considering:
- $\mathbf{q}$ as a suitably κ-deformed Newton-Wigner position operator
- $\mathbf{p}$, $\mathbf{J}$ as the generators of translations and rotations of κ-Poincaré algebra

1-dim:

$$[\mathbf{q}, \mathbf{p}] = i\sqrt{1 \pm \alpha \mathbf{p}^2} \tag{3}$$

$p \in \mathbb{R}$ in the $(+)$-sector
$p \in I(-1/\sqrt{\alpha}, 1/\sqrt{\alpha})$ in the $(-)$-sector

Representation algebra: momentum space

$$\begin{aligned} \mathbf{p}\psi(p) &= p\psi(p) \\ \mathbf{q}\psi(p) &= i\sqrt{1 \pm \alpha p^2}\partial_p \psi(p) \end{aligned} \tag{4}$$

Hilbert spaces $\mathbf{p}$, $\mathbf{q}$ self-adjoint operators:

$$\mathcal{F}_\pm = L^2\left(\mathbb{R}(I), dp/\sqrt{1 \pm \alpha p^2}\right) \tag{5}$$

Hilbert spaces unitarily inequivalent $\Rightarrow$ different physical predictions

For $\alpha \to 0$ the ordinary one $L^2(\mathbb{R}, dp)$ is recovered in both the cases

Harmonic oscillator: $\mathcal{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2$

Considering the above representation (4)

$$(\widehat{\mathcal{H}} - E)\psi(p) = 0 \tag{6}$$

Mathieu equation $\Rightarrow$ $\psi(p)$ in terms of Mathieu cosine and sine

Spectrum at the lowest order ($\sqrt{\alpha}/d \ll 1$):

$$E_n = \frac{\omega}{2}(2n+1) \pm \frac{\omega}{8}(2n^2+2n+1)\left(\frac{\alpha}{d^2}\right) \tag{7}$$

Characteristic length scale $d = 1/\sqrt{m\omega}$

$E_n^{(-)}$ **corresponds to the spectrum of the h.o. in polymer quantum mechanics**

$E_n^{(+)}$ spectrum of GUP ($[\mathbf{q}, \mathbf{p}] = i(1 + \beta\mathbf{p}^2)$)

II. Deformed FRW dynamics

The FRW cosmological models

$$ds^2 = -N^2 dt^2 + a^2 \left(\frac{dr^2}{1 - kr^2} + r^2 d\Omega^2 \right) \quad (8)$$

$N = N(t)$ is the lapse function
$a = a(t)$ the scale factor

Isotropy: phase space of GR 2-dim
$\Gamma = (a, p_a)$

Scalar constraint:

$$\mathcal{H} = -\frac{2\pi G}{3}\frac{p_a^2}{a} - \frac{3}{8\pi G} ak + a^3 \rho = 0 \qquad (9)$$

$\rho = \rho(a)$ denotes a generic energy density

$\rho \sim 1/a^4$ ultra-relativistic gas
$\rho \sim 1/a^5$ perfect gas
$\rho \sim const$ cosmological constant

Extended Hamiltonian:

$$\mathcal{H}_E = \frac{2\pi G}{3} N \frac{p_a^2}{a} + \frac{3}{8\pi G} N a k - N a^3 \rho + \lambda \pi \quad (10)$$

λ is a Lagrange multiplier

π momenta conjugate to N, i.e. it vanishes

Equations of motion:

$$\dot{N} = \{N, \mathcal{H}_E\} = \lambda, \quad \dot{\pi} = \{\pi, \mathcal{H}_E\} = \mathcal{H} \quad (11)$$

Primary constraint $\pi = 0$ satisfied at all times

$\Rightarrow$ scalar (secondary) constraint $\dot{\pi} = \mathcal{H} = 0$

Dynamics of coordinate a and momenta p_a depends on the symplectic structure

If $\{a, p_a\} = 1$ we obtain Friedmann equation

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (12)$$

Modified symplectic geometry

Parameter α independent with respect $\hbar$

$$-i[\mathbf{q},\mathbf{p}] \Longrightarrow \{q,p\}_\alpha = \sqrt{1\pm\alpha p^2} \tag{13}$$

Deformed Poisson bracket:

$$\{F,G\}_\alpha = \{F,G\}\{q,p\}_\alpha = \{F,G\}\sqrt{1\pm\alpha p^2} \tag{14}$$

This is anti-symmetric, bilinear and satisfies the Leibniz rules as well as the Jacobi identity

Time evolution of q, p with respect to $\mathcal{H}$

$$\begin{aligned} \dot{q} &= \{q,\mathcal{H}\}_\alpha = \frac{\partial\mathcal{H}}{\partial p}\sqrt{1\pm\alpha p^2} \\ \dot{p} &= \{p,\mathcal{H}\}_\alpha = -\frac{\partial\mathcal{H}}{\partial q}\sqrt{1\pm\alpha p^2} \end{aligned} \tag{15}$$

FRW deformed phase space

Fundamental commutator:

$$\{a, p_a\}_\alpha = \sqrt{1 \pm \alpha p_a^2} \tag{16}$$

Equation of motion with respect to $\mathcal{H}_E$

$$\dot{a} = \{a, \mathcal{H}_E\}_\alpha = \frac{4\pi G}{3} N \frac{p_a}{a} \sqrt{1 \pm \alpha p_a^2} \tag{17}$$

$$\begin{aligned} \dot{p}_a = \{p_a, \mathcal{H}_E\}_\alpha = N \Bigg(& \frac{2\pi G}{3} \frac{p_a^2}{a^2} - \frac{3}{8\pi G} k + \\ & + 3a^2 \rho + a^3 \frac{d\rho}{da} \Bigg) \sqrt{1 \pm \alpha p_a^2} \end{aligned} \tag{18}$$

Deformed Friedmann equation

$$H^2 = \left(\frac{8\pi G}{3} \rho - \frac{k}{a^2} \right) \left[1 \pm \frac{3\alpha}{2\pi G} a^2 \left(a^2 \rho - \frac{3}{8\pi G} k \right) \right] \tag{19}$$

FRW flat ($k = 0$) model

$$H_{k=0}^2 = \frac{8\pi G}{3}\rho\left(1 \pm \frac{\rho}{\rho_{crit}}\right) \qquad (20)$$

critical density $\rho_{crit} = (2\pi G/3\alpha)\rho_P$.

We have assumed the existence of a fundamental scale, i.e. $\rho \leq \rho_{crit}$

For $\alpha \to 0$, $\rho_{crit} \to \infty \Rightarrow$ ordinary behavior

($-$)-**equation equivalent to LQC**

($+$)-**equation equivalent to braneworlds**

The ($-$) sign implies a bouncing cosmology, while with the ($+$) one $\dot{a}$ can not vanish

A ($-$)-braneworlds scenario appears if the extra-dimension is time-like (open question)

III. Deformed uncertainty principle

Uncertainty principle related to (16)

$$\Delta a = \frac{1}{2}\left|\left(\frac{1 \pm \alpha\langle \mathbf{p}_a\rangle^2}{(\Delta p_a)^2} \pm \alpha\right)^{1/2}\right| \qquad (21)$$

For $\Delta p_a \gg (\Delta p_a)^\star \equiv \sqrt{(1 \pm \alpha\langle \mathbf{p}_a\rangle)/\alpha}$
minimal uncertainty in the scale factor
$\Delta a_0 = \sqrt{\alpha}/2$

Brane-framework, $(+)$-**sector**: $\Delta a_0 \neq 0$ is a global minimum (No physical states which are position eigenstates exist at all)

LQC-framework, $(-)$-**sector**: $\Delta a_0 = 0$ appears for $\Delta p_a = (\Delta p_a)^\star \propto 1/\sqrt{\alpha}$, i.e. when the deformation energy is reached

IV. Discussion and conclusions

- A unique framework which phenomenologically describes both the effective Friedmann evolution of LQC and branewords models is obtained by the use of a deformed Heisenberg algebra.

- The algebra leaves undeformed the translation group and preserves the rotational invariance. Furthermore is related to the κ-Poincaré one and no sign in the deformation term is selected at all.

- The brane-deformed scenario is such that a minimal uncertainty in the scale factor appears. On the other hand, in the loop one, we have the vanishing uncertainty when the deformed energy is reached.