Quantum Cosmology in the GUP Approach

Motivations:
Implement the notion of a fundamental scale in the Universe dynamics. The application of a generalized uncertainty principle (a natural way to realize this intuition) to the minisuperspace dynamics is physically grounded. This way, some features of a more general theory are implemented in quantizing a cosmological model.

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gr-qc/0703025, 0707.2726, 0709.4610

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Summary of talk

I. Quantum mechanics in the GUP scheme

II. GUP in the minisuperspace dynamics

III. Quantum singularity

IV. FRW singularity in the WDW approach

V. FRW singularity in the GUP approach

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VII. Bianchi IX model in the GUP approach

VIII. Discussion and conclusions

I. Quantum mechanics in the GUP scheme

Generalized Uncertainty Principle (GUP)

$$\Delta q \Delta p \geq \frac{1}{2}\left(1 + \beta(\Delta p)^2 + \beta\langle \mathbf{p}\rangle^2\right) \qquad (1)$$

String theory $\Rightarrow$ GUP

GUP can be recovered by a deformed Heisenberg algebra

$$[\mathbf{q}, \mathbf{p}] = i(1 + \beta \mathbf{p}^2) \qquad (2)$$

Nonzero minimal uncertainty in position

$$\Delta q_{min} = \sqrt{\beta} > 0 \qquad (3)$$

This way, GUP $\Rightarrow$ modification canonical quantization prescription via (2)

Eigenstate observable A *implies* $\Delta A = 0$

Position eigenstate can be approximated (to arbitrary precision)

$$\lim_{n\to\infty} (\Delta q)_{|\psi_n\rangle} = \lim_{n\to\infty} \langle\psi|(\mathbf{q} - \langle\mathbf{q}\rangle)^2|\psi\rangle = 0. \tag{4}$$

$|\psi_n\rangle \in D \subset \mathcal{H}$, D dense domain

GUP framework this is *no more* possible

$$\lim_{n\to\infty} (\Delta q)_{|\psi_n\rangle} \geq \Delta q_{min} \tag{5}$$

No physical states which are position eigenstates exist at all

Is possible to construct position eigenvectors, but they are only formal eigenvectors

Information of position:
quasiposition wave function

Representation algebra: momentum space

$$\mathbf{p}\psi(p) = p\psi(p) \quad (6)$$
$$\mathbf{q}\psi(p) = i(1+\beta p^2)\partial_p\psi(p). \quad (7)$$

Maximal localization states $|\psi_\zeta^{ml}\rangle$ proprieties

$$\langle\psi_\zeta^{ml}|\mathbf{q}|\psi_\zeta^{ml}\rangle = \zeta, \quad (\Delta q)_{|\psi_\zeta^{ml}\rangle} = \Delta q_{min} \quad (8)$$

Maximal localization because they obey

$$\Delta q \Delta p = \frac{1}{2}|\langle[\mathbf{q},\mathbf{p}]\rangle| \quad (9)$$

Project arbitrary state $|\psi\rangle$ on $|\psi_\zeta^{ml}\rangle$:
quasiposition wave function $\psi(\zeta) \equiv \langle\psi_\zeta^{ml}|\psi\rangle$

$$\psi(\zeta) \sim \int \frac{dp}{(1+\beta p^2)^{3/2}} e^{i\frac{\zeta}{\sqrt{\beta}}\tan^{-1}(\sqrt{\beta}p)}\psi(p). \quad (10)$$

$\beta = 0 \Rightarrow$ ordinary wave function $\psi(\zeta) = \langle\zeta|\psi\rangle$

II. GUP in the minisuperspace dynamics

Minisuperspace reduction of the dynamics (truncation of the phase space of GR):
field theory → mechanical system

- Homogeneous sector (Bianchi models): **3** degrees of freedom (a_1, a_2, a_3)

- Isotropic sector (FRW models): **1** degree of freedom (a)

Universe dynamics → motion of a particle (in some potential) in a given background

⇒ quantization of a minisuperpsace model in the GUP framework is physically well grounded
Introduction of a minimal lenght in the cosmological dynamics

III. Quantum singularity

Classical singularity: criteria
Global: geodesic incompleteness
Local: divergence scalars (unsatisfactory)

Quantum singularity avoidance:
no rigorous criteria yet

- DeWitt proposal: $\Psi(a=0)=0$
  no physical singularity removal

- Evolution of wave packets:
  no fall in the singularity

- Probabilistic: $P_\delta \equiv \int_0^\delta |\Psi(a,t)|^2 da \simeq 0$
  nocollapse situation

IV. FRW singularity in the WDW approach

FRW flat ($k = 0$) with massless scalar field

$$H_{grav} + H_\phi \equiv -9\kappa p_x^2 x + \frac{3}{8\pi}\frac{p_\phi^2}{x} \approx 0 \quad x \equiv a^3, \tag{11}$$

Phase space 4-dimensional: $(x, p_x; \phi, p_\phi)$

p_ϕ constant of motion $\Rightarrow$ classical trajectory in (x, ϕ)-plane: ϕ internal time

Classical trajectories

$$\phi = \pm\frac{1}{\sqrt{24\pi\kappa}} \ln\left|\frac{x}{x_0}\right| + \phi_0, \tag{12}$$

Singularity ($a = 0$) for $\phi = \pm\infty$

All classical solutions meet the singularity

Quantum level

Canonical quantization ($\beta = 0$) of (11) leads to the WDW equation

$$(\partial_\phi^2 + \widehat{\Theta})\Psi = 0, \quad \Theta \equiv 24\pi\kappa(x p_x^2 x) \qquad (13)$$

$\widehat{\Theta}$ is a self-adjoint operator
(in $\mathcal{H} = L^2(\mathbb{R}_+, dx)$, $\widehat{p}_x$ is no self-adjoint)

Wave function $\Psi(x, \phi)$ evolves as ϕ changes:
ϕ as an *emergent time* for the evolution

Decomposition into positive/negative frequency

$$\Psi_\epsilon(x, \phi) = x^{-1/2} \left(A x^{-i\gamma} + B x^{i\gamma} \right) e^{i\sqrt{24\pi\kappa}\epsilon\phi}, \qquad (14)$$

$\gamma = \frac{1}{2}(4\epsilon^2 - 1)^{1/2} \geq 0$
It is of positive frequency: $i\partial_\phi \Psi = -\sqrt{\widehat{\Theta}}\Psi$

i) $\Psi(x = 0, \phi) = \infty \Rightarrow P_\delta = \infty$

ii) Wave packets fall into the singularity

V. FRW singularity in the GUP approach

ϕ "time": (ϕ, p_ϕ) canonically quantized

$$(\partial_\phi^2 + \widehat{\Theta}_{gup})\Psi = 0 \tag{15}$$

Action of $\widehat{\Theta}_{gup}$ on $\Psi(p)$ is

$$\mu^2(1+\mu^2)^2\frac{d^2\Psi}{d\mu^2}+2\mu(1+\mu^2)(1+2\mu^2)\frac{d\Psi}{d\mu}+\epsilon^2\Psi = 0, \tag{16}$$

$\mu \equiv \sqrt{\beta}p$ is a dimensionless parameter

As before frequency decomposition:
$\Psi(p,\phi) = \Psi(p)e^{i\sqrt{24\pi\kappa}\epsilon\phi}$ (positive frequency)

Changes of variables:

1) $\rho \equiv \tan^{-1}\mu$, $\mu \in [0,\infty) \to \rho \in [0,\pi/2]$
2) $\xi \equiv \ln(\sin\rho)$, $\xi \in (-\infty, 0]$

Solution of the eigenvalue problem

$$\Psi_\epsilon(\xi) = Ce^{-\xi(1-\alpha)}\left(1 + be^{2\xi}\right), \qquad (17)$$

where $\alpha = \sqrt{1-\epsilon^2}$ and $b = (1-\alpha)/(1+\alpha)$

Quasiposition wave function (10)

$$\Psi_\epsilon(\zeta) = C\int_{-\infty}^{0} d\xi \exp\left(\xi + i\zeta \tan^{-1}\left(\frac{e^{\xi}}{\sqrt{1-e^{2\xi}}}\right)\right) \times \left[e^{-\xi(1-\alpha)}\left(1 + be^{2\xi}\right)\right] \quad (18)$$

It is the probability amplitude for the particle (Universe) being maximally localized around the position $\zeta = \langle \psi_\zeta^{ml} | (x = a^3) | \psi_\zeta^{ml} \rangle$

$$\Rightarrow |\Psi_\epsilon(\zeta)| < \infty, \forall\, \zeta$$

Wave packets

$$\Psi(\zeta,t)=\int_0^\infty d\epsilon g(\epsilon)\Psi_\epsilon(\zeta)e^{i\epsilon t} \tag{19}$$

$t=\sqrt{24\pi\kappa}\phi$ is a dimensionless time

$$g(\epsilon)=e^{-\frac{(\epsilon-\epsilon^*)^2}{2\sigma^2}} \tag{20}$$

$\epsilon^*\ll 1$: (we recall that $\epsilon\sim\mathcal{O}(\bar{\epsilon}\ l_P)$)
Gaussian peaked at energy much less then the Plank energy $1/l_P$

In the next we will compute the above integral numerically ($\epsilon^*=10^{-3}$, $\sigma^2=1/20$)

a) $\zeta\simeq 0$: purely Planckian region. From equation (18) one obtain

$$|\Psi(\zeta,t)|^2\simeq|A(t)|^2+\zeta^2|B(t)|^2 \tag{21}$$

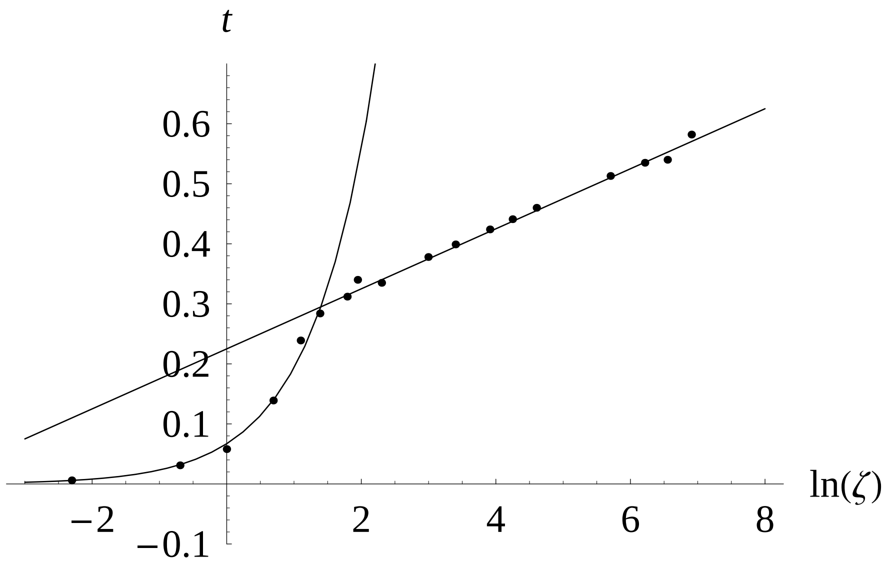
$A(t)$ such that probability density vanishes for $t\to-\infty$, where the classical singularity appears

b) Evolution of the wave packets

The probability density $|\Psi(\zeta,t)|^2$ is well approximated by a Lorentzian function $\forall\,\zeta$

The peaks of Lorentzian move along the classically expanding trajectory (12) for $\zeta \geq 4$

They follow a power-law up to $\zeta = 0$:
escape from the classical trajectory toward the classical singularity



The GUP wave packets do not fall in the singularity, but they approach the Planckian region in a stationary way

FRW ($k = 0$) Universe: WDW vs GUP

- $|\Psi_{WDW}(a = 0)| = \infty$
  Probability $P_\delta \equiv \int_0^\delta |\Psi_{WDW}(a,t)|^2 da = \infty$

  $|\Psi_{GUP}(\zeta = 0)| < \infty$
  Probability $P_\delta \equiv \int_0^\delta |\Psi_{GUP}(\zeta,t)|^2 d\zeta = 0$

- WDW wave packets fall in the classical singularity

  GUP wave packets **do not** fall in the classical singularity. GUP Universe exhibit a stationary behavior in approaching the Planckian region

However in the GUP dynamics there is no a Big-Bounce like predicted in LQC:
the cutoff length does not seems to be the mechanism behind the bounce

VI. Taub model in the GUP approach

Taub cosmological model: particular case of Bianchi IX ($\gamma_- = 0$)

$$ds^2 = N^2 dt^2 - e^{2\alpha} \left(e^{2\gamma}\right)_{ij} \omega^i \otimes \omega^j \qquad (22)$$

$\alpha = \alpha(t)$: isotropic expansion of the Universe

$\gamma_{ij} = \gamma_{ij}(t)$: traceless symmetric matrix which determines the anisotropies *via* $\gamma_\pm$

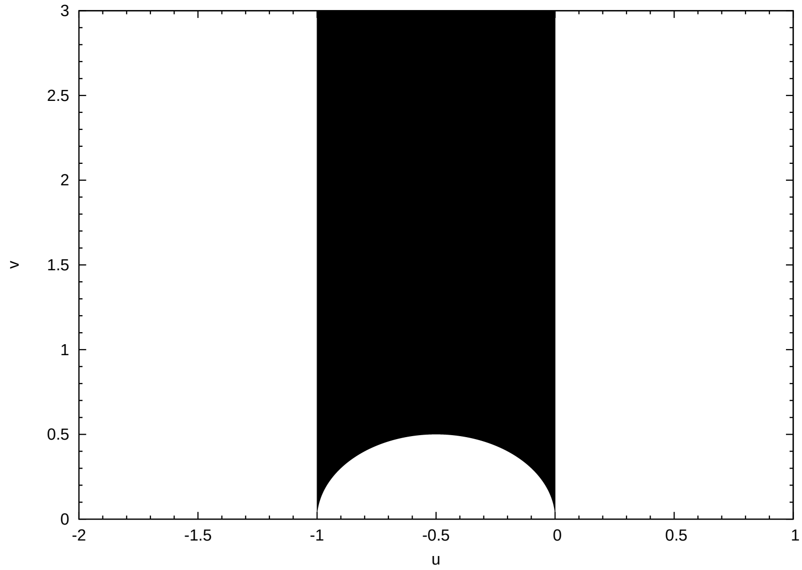
Bianchi IX toward the singularity:
particle in a two-dimensional closed-domain

Allowed-domain:
i) dependent on *time*-variable (Misner)
ii) independent on *time*-variable (Misner-Chitré)

ADM reduction of the dynamics:
$H = 0$ classically

In the Poincaré plane

$$-p_\tau \equiv H^{IX}_{ADM} = v\sqrt{p_u^2 + p_v^2}. \qquad (23)$$



Dynamical-allowed domain

Taub model: $u = -1/2$ ($v \in [1/2, \infty)$)

$$H^{T}_{ADM} = v p_v = p_x \equiv p \qquad (24)$$

$x = \ln v \in [\ln(1/2), \infty)$. Singularity $\tau \rightarrow \infty$

The classical dynamics of this model: massless particle which bounces against a given wall

Classical dynamics

β **independent** with respect $\hbar$

$$-i[\mathbf{q},\mathbf{p}] \Rightarrow \{\mathbf{q},\mathbf{p}\} = (1+\beta p^2) \tag{25}$$

Deformed equation of motion

$$\dot{q} = \{q, H\} = (1+\beta p^2)\frac{\partial H}{\partial p} \tag{26}$$

$$\dot{p} = \{p, H\} = -(1+\beta p^2)\frac{\partial H}{\partial q} \tag{27}$$

Applying to our Hamiltonian (24)

$$x(\tau) = (1+\beta A^2)\tau + cost, \quad p(\tau) = cost = A \tag{28}$$

Angular coefficient is $(1+\beta A^2) > 1$ for $\beta \neq 0$

Incoming trajectory (for $\tau < 0$) will be **closer**, for $\beta \neq 0$, to the outgoing trajectory ($\tau > 0$)

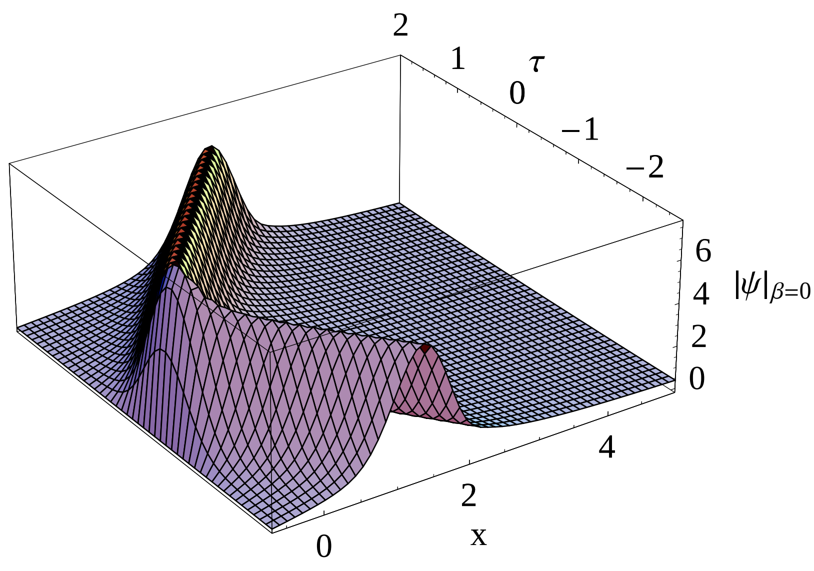
Quantum evolution in the WDW scheme

i) Time variable: canonical quantization
ii) Physical variable x: GUP quantization (we recover the WDW formalism for $\beta = 0$)

x related to the Universe anisotropy β_+ via

$$\beta_+ = \frac{e^{\tau - x}}{\sqrt{3}} \left(e^{2x} - \frac{3}{4} \right) \qquad (29)$$

Evolution of wave packets in the (τ, x)-plane

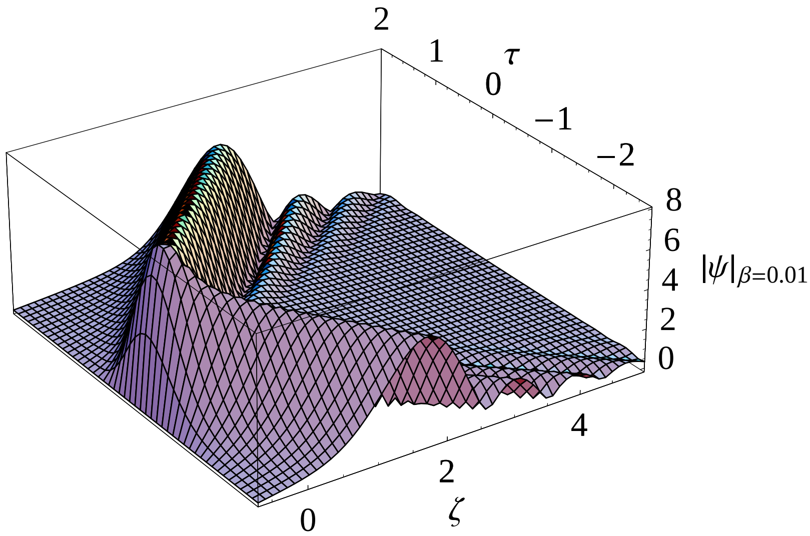


Wave packets peaked around classical trajectories. Universe bounce on the potential wall ($x = \ln(1/2)$) and the fall into the singularity

Quantum evolution in the GUP scheme

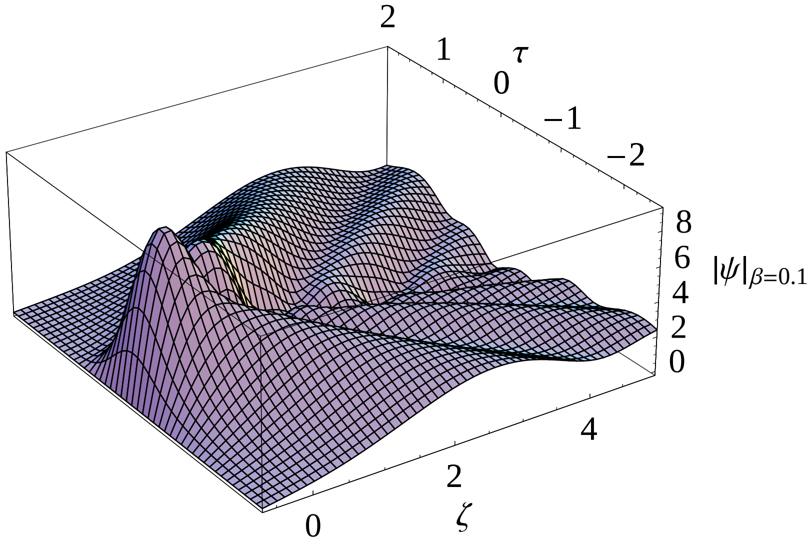
The parameter β is responsible for the GUP effects on the dynamics.
In our case $\Delta x_{min} = \sqrt{\beta}$

- $\beta \sim \mathcal{O}(10^{-2})$. A constructive and destructive **interference** between the incoming and outgoing wave appears



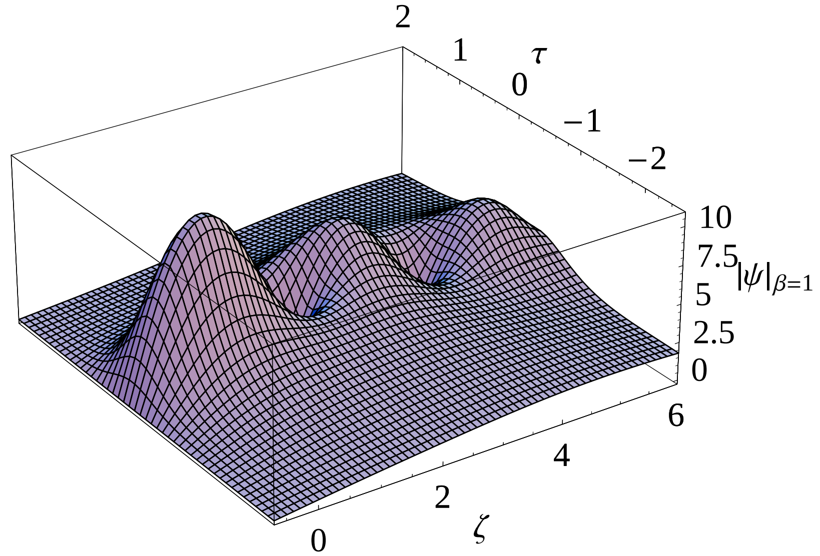
The probability amplitude to find the Universe is still peaked around the classical trajectory, but "not so much" as in the canonical case

- $\beta \sim \mathcal{O}(10^{-1})$. It is **no more possible to distinguish** an incoming or outgoing wave packet



At this level is meaningless to speak about a wave packet which follows the classical trajectory

- $\beta \sim \mathcal{O}(1)$. A dominant probability peak "near" the potential wall appears and the motion of wave packets show a **stationary behavior**, i.e. independent on τ



a) The classical singularity ($\tau \to \infty$) is widely probabilistically suppressed, because the probability to find the Universe is peaked just around the potential wall

b) The large anisotropy states, i.e. those for $|\gamma_+| \gg 1$ ($|\zeta| \gg 1$), are not privileged

VII. Bianchi IX model in the GUP approach

Deformed phase space algebra

$$\begin{aligned}\{q_i, p_j\} &= \delta_{ij}(1+\beta p^2) + \beta' p_i p_j \\ \{p_i, p_j\} &= 0 \\ \{q_i, q_j\} &= (2\beta - \beta') + (2\beta + \beta')\beta p^2 J_{ij}\end{aligned} \quad (30)$$

where $J_{ij} = (1+\beta p^2)^{-1}(p_i q_j - p_j q_i)$

Poisson bracket for any phase space function

$$\{F, G\} = \left(\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_j} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_j}\right)\{q_i, p_j\} + \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial q_j}\{q_i, q_j\} \quad (31)$$

For $\beta = 2\beta'$, $\{q_i, q_j\} = 0 + \mathcal{O}(\beta^2)$ and the isotropic minimal uncertainty in position reads $\Delta q_0 = 2\sqrt{\beta}$

ADM Hamiltonian of Bianchi models

$$-p_\alpha = \mathcal{H} = \left(p_+^2 + p_-^2 + e^{4\alpha} V(\gamma_\pm)\right)^{1/2} \quad (32)$$

Cosmological singularity for $\alpha \to -\infty$

The differences between the models are summarized in the potential term $V(\beta_\pm)$

By the ADM scheme, α is the time-coordinate and thus the deformed equation of motion:

$$\dot{\gamma}_i = \{\gamma_i, \mathcal{H}\} = \frac{1}{\mathcal{H}}\left[(1+\beta p^2)\delta_{ij} + 2\beta p_i p_j\right] p_j \quad (33)$$

$$\dot{p}_i = \{p_i, \mathcal{H}\} = -\frac{e^{4\alpha}}{2\mathcal{H}}\left[(1+\beta p^2)\delta_{ij} + 2\beta p_i p_j\right]\frac{\partial V}{\partial \gamma_j}$$

The deformed Bianchi I model

For Bianchi I: $V(\gamma_\pm) = 0$. Is described by a two-dimensional massless scalar relativistic particle

Deformed equations of motion (33) are immediately solved by

$$\dot{\gamma}_\pm = C_\pm(\beta) \qquad \dot{p}_\pm = 0 \tag{34}$$

Therefore, the solution is Kasner-like

Velocity of the particle (Universe) becomes

$$\dot{\gamma}^2 = \dot{\gamma}_+^2 + \dot{\gamma}_-^2 = \frac{p^2}{\mathcal{H}^2}\left(1 + 6\mu + 9\mu^2\right) = 1 + 6\mu + 9\mu^2 \tag{35}$$

where $\mu = \beta p^2$

Two Kasner indices can be negative at the same time. **The Universe contracts along one direction while grows along the other two**

The deformed Bianchi II model

Dynamics of a 2-dimensional particle in a plane with a potential wall: $V(\gamma_\pm) = e^{-8\gamma_+}$

$$\mathcal{H} = \left(p_+^2 + p_-^2 + e^{4(\alpha - 2\gamma_+)}\right)^{1/2} \qquad (36)$$

In the undeformed scheme
$\dot{\gamma}_\pm = p_\pm/\mathcal{H} \quad \Rightarrow \dot{\mathcal{H}} = 0$ for $\alpha \to -\infty$

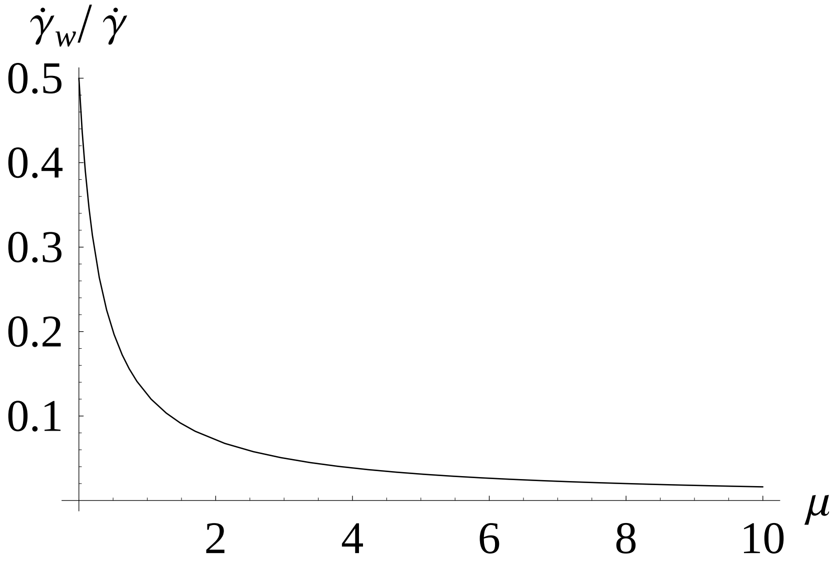
In the deformed scheme $\dot{\mathcal{H}} \neq 0$

Velocity of the particle (Universe)
$\dot{\gamma}^2 = 1 + 6\mu + 9\mu^2$

Potential term important if $e^{4(\alpha-2\gamma_+)} \simeq \mathcal{H}^2$
Wall velocity $\dot{\gamma}_w$

$$\dot{\gamma}_+ \simeq \dot{\gamma}_w = \frac{1}{2} - \frac{1}{8}\frac{\partial}{\partial\alpha}(\ln \mathcal{H}^2) = \frac{p^2(\dot{\gamma})}{2\mathcal{H}^2} \qquad (37)$$

$\dot{\gamma}_w$ and $\dot{\gamma}$ in function of $\mu = \beta p^2$



Parametrization of $\dot{\gamma}$:
initial state $(\dot{\gamma}_+)_i = -\dot{\gamma}\cos\theta_i$, $(\dot{\gamma}_-)_i = \dot{\gamma}\sin\theta_i$
final state $(\dot{\gamma}_+)_f = \dot{\gamma}\cos\theta_f$, $(\dot{\gamma}_-)_f = \dot{\gamma}\sin\theta_f$

Maximum angle: $|\theta_i| < |\theta_{\rm max}| = \cos^{-1}(\dot{\gamma}_w/\dot{\gamma})$
$|\theta_{\rm max}| = \pi/2$ for $\mu \gg 1$
$|\theta_{\rm max}| = \pi/3$ ordinary case $(\dot{\gamma}_w/\dot{\gamma} = 1/2)$

Deformed Bianchi II is no longer analytically integrable: no BKL map can in general computed

Deformed Mixmaster Universe

Particle in a 2-plane which bounces infinites times against a triangular domain (chaotic dynamics)

$V(\gamma_\pm) = e^{4(\gamma_+ + \sqrt{3}\gamma_-)} + e^{4(\gamma_+ - \sqrt{3}\gamma_-)} + e^{-8\gamma_+}$
and geometry invariant under $SO(3)$

Inside the allowed domain: Bianchi I
Each potential wall described by Bianchi II

Stationary triangular closed domain:
bounces of the particle increased (no longer maximum limit angle appears)

Ordinary Bianchi II (part of a chaotic system) is an integrable model
Deformed framework Bianchi II, still part of Bianchi IX, no longer integrable

Deformations make the model much more complicated and these are not able to cast a chaotic system in a non-chaotic one

VIII. Discussion and conclusions

- Application of the GUP approach to the minisuperspace models is a natural way to implement the notion of a fundamental scale in the cosmological dynamics

- The FRW Universe appear to be singularity-free, but no evidences for a Big-Bounce, as predicted in LQC, arise

- In the Taub model the singularity is probabilistically suppressed and a quasi-isotropic configuration for the Universe is favored

- The Bianchi I dynamics is Kasner-like but is deeply modified. Mixmaster Universe can still regarded as a chaotic system