Relativistic Boltzmann equations for the pair plasma in presence of proton loading

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Pescara July 8–18 2008

Motivation of the work

- We study kinetic properties of γ, $e^{\mp}$, p plasma, relevant, for example, to GRB phenomenon.
- We want to check assumptions used for the investigation of such phenomenon: the hydrodynamical approximation, character timescales, optical depths. For example, for GRB plasma, where are two different viewpoints. First approach, plasma reaches thermal equlibrium due to huge optical depth and after this, it's expand (can be described by hydrodynamic) reaches large Lorenz factor (Ruffuni et al 1999). Second possible scenario proposed by Cavallo and Rees 1978, the plasma cools down due to direct bremsstrahlung process until the temperature becomes below $m_e c^2$, and pairs disappears.
- We consider the uniform and isotropic plasma.
- In our recent publication 2007 we studied γ, $e^{\mp}$ plasma evolution in the frame of kinetic approach.
- Now we want to study such plasma with p loading in the frame of kinetic Bolzmann equations.

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- The plasma parameter $\left(nr_{\mathrm{D}}^3\right)^{-1} \ll 1.$ The plasma can be described by 1-particles distribution functions $f_i\left(t, \mathbf{p}\right)$.
- The plasma is non-degenerate exept the upper bound of the temperatyre region.

The task to be solved

In this talk we consider homogenous and isotropic plasma

$$\frac{1}{c}\frac{\partial f_i(\epsilon,t)}{\partial t} = \sum_q (\eta_i^q - \chi_i^q f_i),$$

where "i" is the particle kind, "q" is the number of the reaction. We take arbitrary intial data $f_i(\epsilon,0)$. We consider the time evolution to the steady state.

1. We want to know is it possible to describe the GRB plasma in the approximation of the thermal equilibrium? (Goodman 1986 proposed.)
2. Are different scenarios for the GRB proposed by Cavallo and Rees 1978 valid?
3. Is the prediction about the relaxation to the thermal equlibrium by Pilla, Shaham 1997 true?
4. Which timescales do we have?

Reactions rates

Textbook Berestetskii et al 1982, Swensson, 1984, Haug, 1985.

Binary interactions	Radiative and pair producing variants
Møller, Bhabha $e_1^{\pm} e_2^{\pm} \longrightarrow e_1^{\pm\prime} e_2^{\pm\prime}$ $e_1^{\pm} e_2^{\mp} \longrightarrow e_1^{\pm\prime} e_2^{\mp\prime}$	Bremsstrahlung $e_1^{\pm} e_2^{\pm} \leftrightarrow e_1^{\pm} e_2^{\pm} \gamma$ $e_1^{\pm} e_2^{\mp} \leftrightarrow e_1^{\pm\prime} e_2^{\mp\prime} \gamma$
Single Compton $e^{\pm}\gamma \longrightarrow e^{\pm}\gamma'$	Double Compton $e^{\pm}\gamma \leftrightarrow e^{\pm\prime}\gamma'\gamma''$
Pair production and annihilation $\gamma\gamma' \leftrightarrow e^{\pm}e^{\mp}$	Radiative pair production and three photon annihilation $\gamma\gamma' \leftrightarrow e^{\pm}e^{\mp}\gamma''$ $e^{\pm}e^{\mp} \leftrightarrow \gamma\gamma'\gamma''$

Table: Reactions with $e^{\mp}$

Reactions rates

Binary interactions	Radiative and pair producing variants
Coulomb scattering	Bremsstrahlung
$p_1 p_2 \longrightarrow p_1' p_2'$	$p_1 p_2 \longleftrightarrow p_1' p_2' \gamma$
$pe^{\mp} \longrightarrow p' e^{\mp\prime}$	$pe^{\mp} \longleftrightarrow p' e^{\mp\prime} \gamma$

Table: Reactions with protons

Numerical Method

1. We introduced the computational grid for the phase space ϵ, μ, ϕ (instead of $\mathbf{p}$). We replaced the integrals by sums. We obtained the set of ODE's to solve.
2. There are several characteristic times for different processes in the problem. The obtained system of ODE's is stiff. (Eigenvalues of Jacobi matrix differs significantly, and the real parts of eigenvalues are negative.) We used Gear's method (Hall & Watt 1976) to integrate ODE's numerically. This high-order stable implicit method. We do not use Monte Carlo simulations (Pilla, Shaham can reach only kinetic equilibrim by such approach).
3. Our code is conservative for the energy. Also the method conserves the particles number. We prefer to use, instead of distribution functions f_i, spectral energy densities

$$E_i(\epsilon) = \frac{4\pi\epsilon^3\beta_i(\epsilon)f_i}{c^3}, \quad \epsilon_i f_i(\mathbf{p},t)d\mathbf{r}d\mathbf{p} = \frac{4\pi\epsilon^3\beta_i f_i}{c^3}d\mathbf{r}d\epsilon_i = E_i d\mathbf{r}d\epsilon. \tag{1}$$

Two particles interactions. $ep \longrightarrow e'p'$

The time evolution of the distribution functions of electrons and protons due to $ep \longrightarrow e'p'$ is described by

$$\frac{\partial f_e(\mathbf{p},t)}{\partial t} = \int d\mathbf{q} d\mathbf{p}' d\mathbf{q}' w_{\mathbf{p}',\mathbf{q}';\mathbf{p},\mathbf{q}} [f_e(\mathbf{p}',t) f_p(\mathbf{q}',t) - f_e(\mathbf{p},t) f_p(\mathbf{q},t)], \quad (2)$$

$$\frac{\partial f_p(\mathbf{q},t)}{\partial t} = \int d\mathbf{p} d\mathbf{p}' d\mathbf{q}' w_{\mathbf{p}',\mathbf{q}';\mathbf{p},\mathbf{q}} [f_e(\mathbf{p}',t) f_p(\mathbf{q}',t) - f_e(\mathbf{p},t) f_p(\mathbf{q},t)], \quad (3)$$

where

$$w_{\mathbf{p}',\mathbf{q}';\mathbf{p},\mathbf{q}} = \frac{c\delta(\epsilon_e + \epsilon_p - \epsilon'_e - \epsilon'_p)}{(2\pi\hbar)^2} \delta(\mathbf{p} + \mathbf{q} - \mathbf{p}' - \mathbf{q}') \frac{|M_{fi}|^2}{16\epsilon_e\epsilon_p\epsilon'_e\epsilon'_p}, \quad (4)$$

$$|M_{fi}|^2 = 2^6(\pi\hbar)^2 e^4 c^3 \frac{\frac{1}{2}(s^2+u^2) + (m^2c^2 + M^2c^2)(2t - m^2c^2 - M^2c^2)}{t^2},$$

the kinematics invariants are $s = (\mathfrak{p} + \mathfrak{q})^2$, $t = (\mathfrak{p} - \mathfrak{p}')^2$, $u = (\mathfrak{p} - \mathfrak{q}')^2$.

$ep \longrightarrow e'p'$ details

$$\chi_e^{ep} = \int do'_e d\mathbf{q} \frac{\beta'_e}{\beta'_e - \beta'_p \mathbf{e}'_p \mathbf{e}'_e} \frac{\epsilon'^2_e \beta'_e}{16\epsilon_e \epsilon_p \epsilon'_e \epsilon'_p} \frac{|M_{fi}|^2}{c^3 (2\pi\hbar)^2} f_p(\mathbf{q}, t),$$

where $do'_e = d\mu'_e d\phi'_e$ is the element of the soliod angle for outgoing electron in the laboratory frame, $\epsilon'_e\left(\mu'_e, \phi'_e\right)$ is it's energy.
Possible solution for the reaction $ep \rightarrow e'p'$ is

$$f_i\left(\epsilon, t\right) \propto \exp\left(\frac{-\epsilon + \varphi_i}{kT}\right), \text{ any } \varphi_i \neq 0.$$

Cutoff of the scattering for little angles

We have infinite integrals at calculations of scattering charged particles due to terms like as t^2 in the denominator of $|M_{fi}|^2$. To prevent it one should take into account Debye screening, and disregard little angles of scattering. Haug 1988 gives the minimal scattering angle in the center mass system

$$\theta_{\min} = \frac{2\hbar(m+M)}{mMcD}\frac{\Gamma}{(\Gamma+1)\sqrt{2(\Gamma-1)}},$$

where the maximum impact parameter D is

$$D^{-2} = \frac{4\pi e^2 n_e/m}{(p_0 c/\epsilon_{01})^2 c^2} + \frac{4\pi e^2 n_p/M}{(p_0 c/\epsilon_{02})^2 c^2},$$

Γ is the invariant Lorentz factor.

Center of mass frame

The Center mass frame definition is

$$\mathbf{p}_{10} + \mathbf{p}_{20} = 0,$$

while the Lorenz transformation for energy-momentum is (CM system moves with $\mathbf{V}$ in Laboratory frame)

$$\mathbf{p}_{i0} = \mathbf{p}_i + \left[(\Gamma - 1)(\mathbf{N}\mathbf{p}_i) - \Gamma V \epsilon_i\right]\mathbf{N}, \;\; i = 1, 2,$$

$$\epsilon_i = \Gamma(\epsilon_{i0} + \mathbf{V}\mathbf{p}_{i0}).$$

Then

$$\mathbf{V} = \frac{\mathbf{p}_1 + \mathbf{p}_2}{\epsilon_1 + \epsilon_2}, \mathbf{N} = \frac{\mathbf{V}}{V}, \;\; \Gamma = \frac{1}{\sqrt{1 - V^2}},$$

$$\mathbf{e}_{10} = -\mathbf{e}_{20}, \;\; \mathbf{e}'_{10} = -\mathbf{e}'_{20},$$

$$\mathrm{abs}(p_{10}) = \mathrm{abs}(p_{20}) = p_0 \equiv \sqrt{\epsilon_{10}^2 - m^2} \equiv \sqrt{\epsilon_{20}^2 - M^2}.$$

Then invariant $t \geq t_{\min}$, there we can calculate invariant in CM frame $t_{\min} = t\left(p_0, \theta_{\min}\right)$.

We can replace t^2 in the denominator of $\left|M_{fi}\right|^2$ by the value $t^2 + t_{\min}^2$ to cutoff the small angles.

Mass scaling for $ep \longrightarrow e'p'$

Let $m, \epsilon_e \ll M$ then $\mathbf{V} \approx \frac{\mathbf{p}_1+\mathbf{p}_2}{M}$, $\Gamma \approx 1$, $\epsilon_1 \approx \left(\epsilon_{01} + \mathbf{V}\mathbf{e}_{01} p_0\right)$,
$\epsilon_1' \approx \left(\epsilon_{01} + \mathbf{V}\mathbf{e}'_{01} p_0\right)$, $\epsilon_1' - \epsilon_1 \approx \mathbf{V}\left(\mathbf{e}'_{01} - \mathbf{e}_{01}\right) p_0 \propto \frac{1}{M}$. $\beta_p \ll 1$,

$$s = m^2 + M^2 + 2mM, \;\; u = m^2 + M^2 - 2mM,$$

$$|M_{fi}|^2 \propto \frac{\frac{1}{2}\left(s^2 + u^2\right) + \left(m^2 + M^2\right)\left(2t - m^2 - M^2\right)}{t^2} \propto \frac{\left(6m^2 - 2t\right) M^2}{t^2},$$

$$\eta^{ep}_{e\omega} - (\chi E)^{ep}_{e\omega} \propto \int \frac{\left(\epsilon_e' - \epsilon_e\right)|M_{fi}|^2}{\epsilon_e \epsilon_p \epsilon_e' \epsilon_p'} \propto \frac{1}{M}.$$

We can calculate $\eta^{ep_0}_{e\omega}$, $(\chi E)^{ep_0}_{e\omega}$ for the particle with mass $M_0 \gg m$, ϵ instead of M and to obtain the transformation

$$\eta^{ep}_{e\omega} \approx \frac{M_0}{M} \eta^{ep_0}_{e\omega}, (\chi E)^{ep}_{e\omega} \approx \frac{M_0}{M} (\chi E)^{ep_0}_{e\omega}.$$

At $t \longrightarrow t_{\min}$ terms $\propto M^{-1}$ can be important!

Three particles interaction. $ee \leftrightarrow e'e'\gamma'$

Calculations of emission and absorption coefficients for triple interactions we illustrate here for bremsstrahlung $e_1 e_2 \leftrightarrow e_1' e_2' \gamma'$. In in the unit of time in the unit volume (in relativistic units $\hbar = 1$, $c = 1$) one has

$$dw = A \prod_{\substack{\text{all particles}\\ \text{on exit } a}} \frac{d\mathbf{p}_a'}{(2\pi)^3},$$

where $A = (2\pi)^4 \delta^{(4)}(P_f - P_i) \frac{|M_{fi}|^2}{\underbrace{2\epsilon_1 ..}_{\substack{\text{all input,}\\ \text{output particles}}}}$, and

$$\dot{f}_2 = \int \frac{dp_1 dp_1' dp_2' dk'}{(2\pi)^6} A \left(f_1' f_2' f_k' - \frac{1}{(2\pi)^3} f_1 f_2 \right).$$

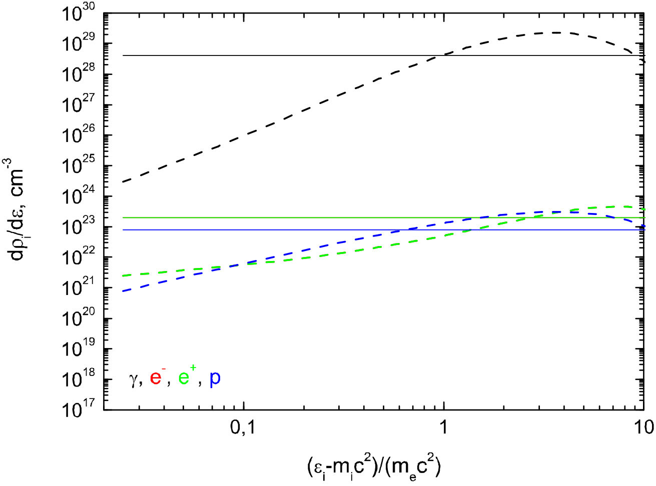
Let's take distribution functions $f_i = \frac{1}{(2\pi)^3} \exp \frac{-\epsilon_i + \varphi_i}{kT}$, we have multipliers proportional to $\exp \frac{\varphi}{kT}$ in front of the integrals.

The calculation of emission and absorption coefficients is then reduced to the well known thermal equilibrium case by Svensson $\chi(\epsilon) = \chi^{\rm eq}$, $\eta \propto \chi^{\rm eq} f$. Such approach is justified since distribution functions with definite chemical potential and temperature are established before triple interactions become important. We notice also that when $\dot{f}_2 = 0$ and $\varphi_\gamma = 0$, the distribution functions fulfill the condition

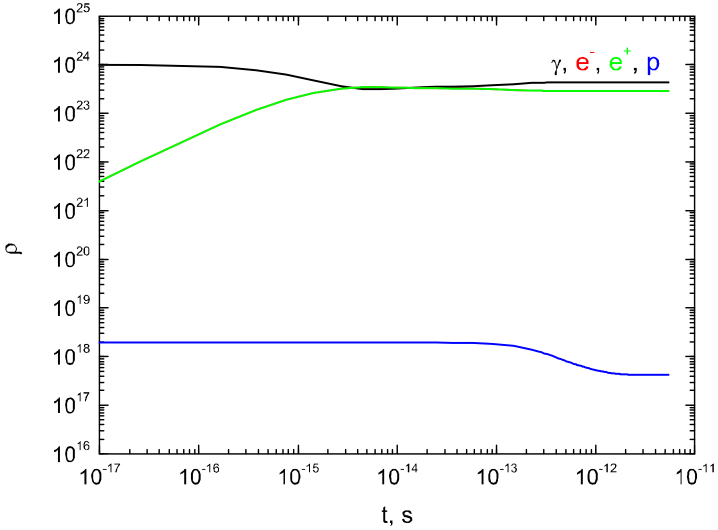
$$f_1 f_2 = f_1' f_2' f_k', \;\; \epsilon_1 + \epsilon_2 = \epsilon_1' + \epsilon_2' + \epsilon_k'.$$

The initial energy spectrum. $t = 0$.

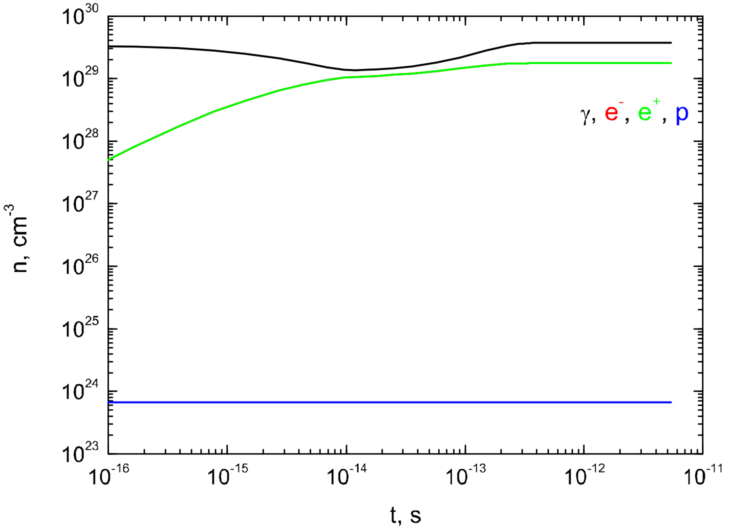
Main part of $\rho = 10^{24}$ ergs cm^{-3} in γ, $B \equiv \frac{n_p M c^2}{\rho} = 10^{-3}$.



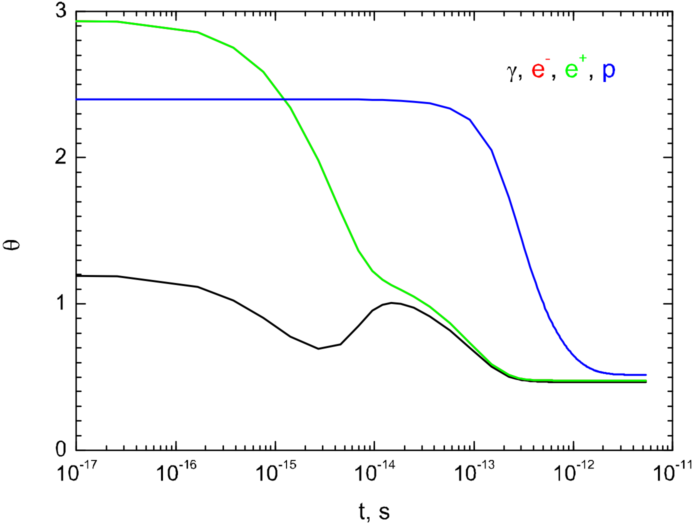
Dependence of energy densities from time



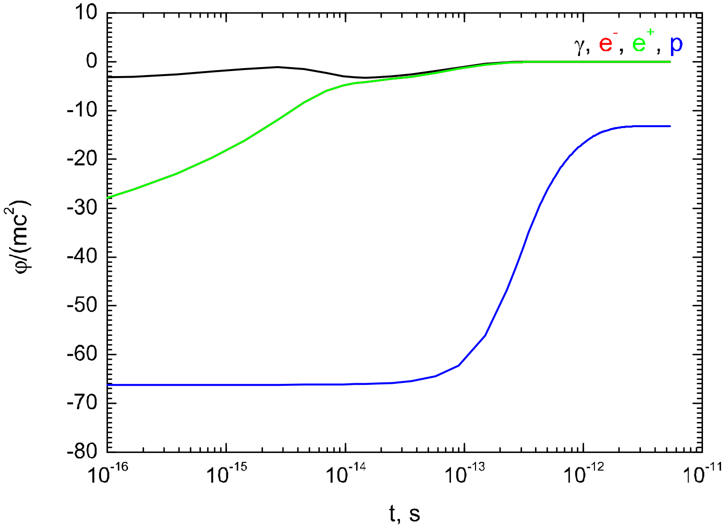
Dependence of concentration from time



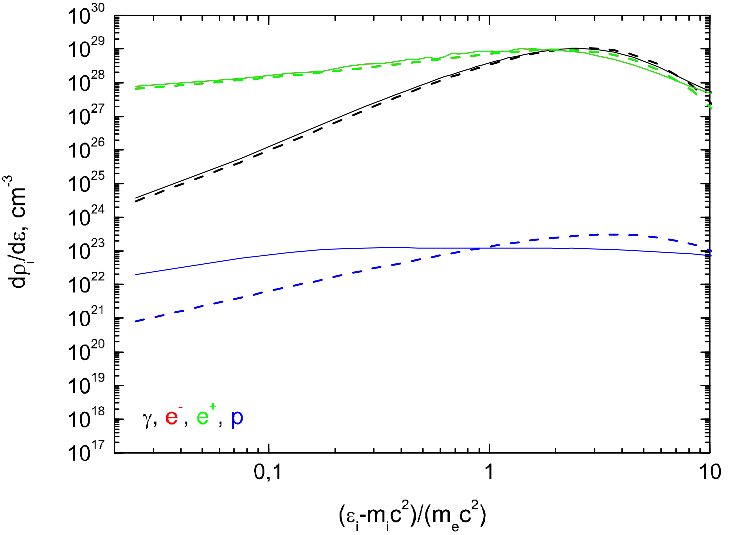
Dependence of temperatures from time



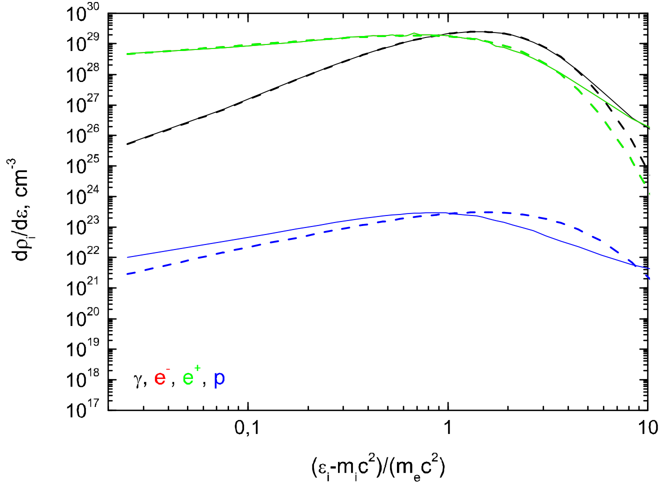
Dependence of chemical potentials from time



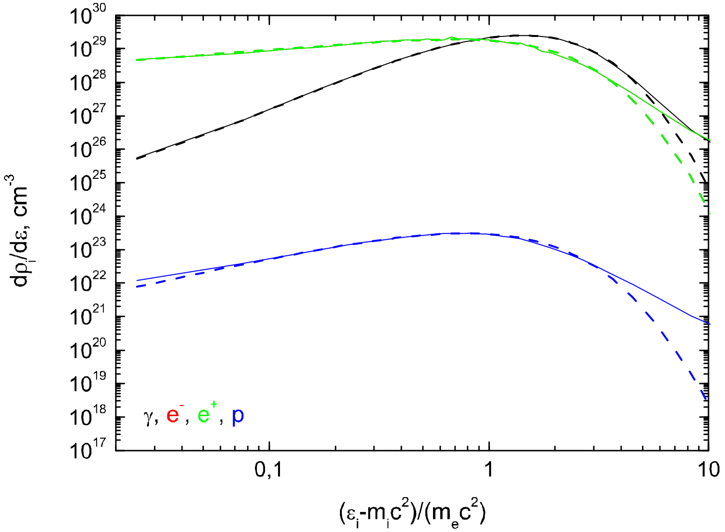
Spectrum at electrons kinetic equilibrium $t = 4 \cdot 10^{-14}$ s



Spectrum at electrons thermal equilibrium fro γ, $e^{\mp}$ $t = 5 \cdot 10^{-13}$ s



Spectrum at thermal equilibrium fro γ, $e^{\mp}$ $t = 5 \cdot 10^{-12}$ s



Conclusions

1. We get two types of equilibriums in $e^{\mp}$, γ plasma from first principles. So called kinetic equilibrium with
$$T,\ \varphi \neq 0,\ \text{timescale} \sim (n\sigma_0 c)^{-1}$$
due to 2 particles reactions. And also thermal equilibrium with T, $\varphi = 0$ due to detailed balance in 3 particles reactions
$$T,\ \varphi = 0,\ \text{timescale} \sim (\alpha n_e \sigma_0 c)^{-1}.$$
For protons we see timescale can be $\sim \frac{M}{m}(n\sigma_0 c)^{-1}$ at low B.
2. The timescale for thermal equilibrium $\lesssim 10^{-12}$ s is much shorter comparing to the timescale of the expansion ($\gtrsim$ msec).
3. We used the approximation of the uniform plasma. This means the results can be used for the real plasma the the character dimension $\gg (n\sigma_{\mathrm{T}}) = 4 \cdot 10^{-5}$ cm.
4. In the non degenerate case we can reduce 3-particles reaction to the equilibrium reactions rates, because we have temperature and chemical potential in the kinetics equilibrium.

Future perspectives

1. Although the kinetic approch proofs the hydrodinamical approach, it can be useful to study the set of thenomena with plasma far from the equlibrium: the study of the decay of the critical electrical field or the investigation of some compact object al low radiation. For example, the radiation a Strange Star ($R = 10$ km) is non-equlibrium even for large luminosities $\lesssim 10^{42}$ ergs s^{-1}. Such can requre to solve nonuniform Bolzmann equations for differents kinds of particles $f_i(\mathbf{r}, \mathbf{p}, t)$.
2. For GRB's plasma it is also can be interesting to consider the kinetic approach for photons to study of spectra.