

Measuring the angular solar diameter using two pinholes

Costantino Sigismondi^{a)}

International Center for Relativistic Astrophysics, Department of Physics, University of Rome "La Sapienza," P. le A. Moro, 2 00185 Rome, Italy and Department of Astronomy, Yale University, 260 Whitney Avenue, New Haven, Connecticut 06511

(Received 18 February 2002; accepted 23 July 2002)

At the turn of the sixteenth century Tycho Brahe and Johannes Kepler made single pinhole measurements of the solar diameter. Their accuracy was limited by diffraction (unknown to them) and the motion of the image on the screen. We discuss how two pinholes built on the same mask can be used to bypass all the problems inherent in the single pinhole approach. The distance at which the two images of the Sun are in contact is the only measurement needed, and the experimental accuracy is much better than measuring the diameter of a single moving image. We obtained 0.5% accuracy, sufficient to follow the angular variations of the solar diameter due to the motion of the Earth in its orbit. © 2002 American Association of Physics Teachers.

[DOI: 10.1119/1.1507790]

I. INTRODUCTION AND DEFINITION OF INSTRUMENT PARAMETERS

Measurements of the angular solar diameter using a single pinhole were performed by Tycho Brahe¹ in 1591 and Johannes Kepler² in 1600–1602. They calculated the ratio

$$\theta_{\odot} = (d_m - d_p)/f, \quad (1)$$

where d_m is the diameter of the image on the screen, d_p is the diameter of the pinhole, and f is the focal length. The latter is the distance between the center of the image on the screen S and the center of the pinhole, mounted on a mask M parallel to the screen (see Fig. 1). The mean angular solar diameter is 32 arcmin (1920 arcsec), which produces pinhole images whose diameter is about 1/100 of the focal length. The motion of the image during the measurements due to the rotation of the Earth and the low contrast of the edges of the solar image on the screen are causes of errors in the evaluation of the angular solar diameter. In previous work we discussed the original data of Kepler and Tycho, and explained the systematic errors in their results in terms of geometrical and diffraction effects.³

Because Eq. (1) does not take into account effects due to diffraction, Kepler adopted various ways to improve his measurements. (1) He pre-drew disks of different diameters on the screen S where the Sun's image was projected, in order to compare quickly the solar image with the disks. In this way he limited the error due to the motion of the image. (2) He arranged his instrument (Fig. 1) into a *camera obscura* for enhancing the contrast of the image.⁴ (3) By using Eq. (1) to analyze his data, Kepler applied the concepts of geometrical optics, first developed by him.² In effect, he considered that geometrical effects occur when the pinhole is close to the screen where the image is projected: the shape and the dimension of the image are still influenced by the shape and the dimensions of the pinhole, d_p . In the limit $f \rightarrow 0$, $d_m \rightarrow d_p$. An ideal point-like pinhole would produce an image of the Sun with diameter $d_m \rightarrow 0$ when $f \rightarrow 0$. In this way the ratio $\theta_{\odot} = d_m/f$ would be independent of the focal length. But in real experiments the ratio $\theta_{\odot} = d_m/f$ tends to a constant value only at large focal lengths. Kepler subtracted the value of d_p from d_m to recover the effect of an ideal point-like pinhole, for which Eq. (1) is valid in the geometri-

cal optics limit. Conversely, diffraction effects, unknown to Kepler, enlarge the image size d_m by a fixed angular amount, so they become important at large focal distances. Due to diffraction, the ratio $\theta_{\odot} = d_m/f$ is systematically larger than the ideal point-like pinhole for large focal distances.

II. IMPROVING THE SINGLE-PINHOLE MEASUREMENTS

The dimensions of the camera obscura where Kepler did his measurements limited the focal length of the instrument shown in Fig. 1 ($f \leq 4$ m). We use a mirror that projects the sunlight horizontally on a pinhole, oriented toward a white screen located in a convenient shadowed place. With this method it is possible to easily obtain good images at $f \sim 20$ m near the best focus. The mirror can be aimed into a real camera obscura, or it can be replaced by a shadowed white screen (or wall), which is easy to find. The pinhole-shadowed screen can be considered a very large pinhole camera.

The existence of an optimal focal length can be understood by the following argument. The geometrical distortion becomes significant for short focal lengths, when the dimension of the solar image approaches d_p ; therefore, the diameter of the image has to be $d_m \gg d_p$. Similarly, diffraction affects the resolution of features on the solar surface that are smaller than an angular distance $\theta_{\text{Rayleigh}} = 1.22\lambda/d_p$, which corresponds to a dimension on the focal plane of $r = 1.22f\lambda/d_p$. If we increase f and d_m , the same information (in terms of resolution) is spread out over a larger surface, thereby losing contrast. Hence d_m cannot be arbitrarily large. Therefore the best focus, f_{best} , is obtained when $r \sim d_p$, and therefore

$$f_{\text{best}} \sim 1/\lambda d_p^2. \quad (2)$$

A pinhole of diameter $d_p = 4$ mm has $f_{\text{best}} = 16$ m for $\lambda = 5500 \text{ \AA}$.

As the pinhole camera gets longer, its magnification increases making the image larger. But as the camera gets longer, the light passing through the pinhole starts to spread due to diffraction. Geometrical effects dominate for $f < f_{\text{best}}$ and diffraction effects for $f > f_{\text{best}}$, so measurements are best

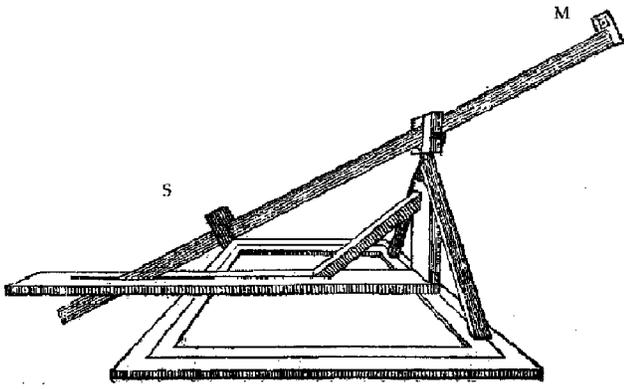


Fig. 1. Kepler's instrument for measuring the solar diameter (Ref. 2). It was 4 m long and it was used in a camera obscura.

made at $f=f_{\text{best}}$. Furthermore, the best focus given by Eq. (2) corresponds to the boundary between the Fresnel diffraction regime and the Fraunhofer diffraction regime.

The ability of the observer to measure the diameter of the moving image also determines the accuracy of the measurement. The rotation of the Earth causes the image to move at about 15 arcsec/s or 1.2 mm/s at a focal length of 16 m. It imposes a severe limit on the accuracy of the measurements of the solar diameter, even at the best focus. The comparison with drawn disks has to be made in less than a second, which makes this measurement a challenging task.

III. TWO PINHOLES

Our goal is to use two pinholes built on the same mask, instead of, for example, giving an hourly motion to the whole instrument to keep it aimed at the Sun. The two pinholes are separated by a fixed distance d . A flat mirror projects onto it the light of the Sun. The pinholes produce two images of the Sun on a screen that is parallel to the mask. The centers of these images are also separated by d . If we slightly vary the focal length f , we can find the focus f_c at which the two images are in contact. The uncertainty in the resolution of the image due to diffraction is now transferred to the determination of f_c . In this way we overcome the uncertainty arising from measuring the position of two moving limbs (edges) of the solar image, and need to measure only the focal length f_c when the contact of the two images is observed.

Because geometrical effects enlarge the images by a fixed amount d_p (see the focal plane of Fig. 2), the actual solar diameter as produced by an ideal point-like pinhole is now given by

$$\theta_{\odot} = (d - d_p) / f_c. \quad (3)$$

Equation (3) is similar to Eq. (1), the difference is that the separation of the centers d and the diameter of the pinholes d_p are quantities that are measurable with the accuracy possible in the laboratory.

We can also work with $f_c \ll f_{\text{best}}$ and low contrast images, but the best results are obtained at $f_c = f_{\text{best}}$, where diffraction effects are minimized. The possibility of measuring only f_c during the observations of the Sun, allows us to obtain an accuracy of measurement of ~ 1 cm over $f_c \sim 5$ m of focal length, if the 4-mm pinholes are mounted 5 cm apart from the other. The distance of the centers of the pinholes can be

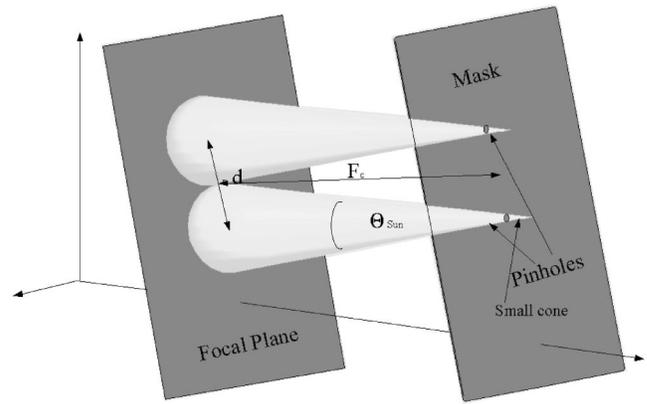


Fig. 2. Two-pinhole geometry. The solar light is coming from the right-hand side. The mask (which contains the pinholes) and the focal plane (where the screen is located) are parallel to each other. An ideal pinhole (point-like) in geometrical optics would produce a perfect cone of solar light with an opening angle of Θ_{Sun} . In real pinholes each point of the hole is like an ideal pinhole producing a cone. The cones represent the envelope of the cones produced by each point in each hole. The axes of the ideal cones are parallel to each other, because the Sun is practically at infinity. Consequently the opening angle of ideal cones is equal to the opening angle of their envelope. They also represent the solar rays forming the two images of the Sun on the focal plane. The diameter of the circle at the focal plane is given by $d = \Theta_{\text{Sun}} \cdot (F_c + h)$ where h is the height of the small cone (above the mask) whose base of diameter d_p is the pinhole itself. The proportion $h : (F_c + h) = d_p : d$, from which Eq. (3) is derived, is always valid in the geometrical optics regime.

determined with an accuracy of $\sim 1/10$ mm and their diameter with an accuracy of $\sim 1/100$ mm in the laboratory. With such a compact device an accuracy of $\Delta\theta_{\odot} / \theta_{\odot} \sim 5 \times 10^{-3}$ ~ 10 arcsec is expected from Eq. (3) for the evaluation of the actual angular solar diameter. Finally, instead of its actual value, we might want to evaluate only the changes of the angular solar diameter during the year. Using the same device for this purpose, the uncertainties on the dimensions of d_p and d are systematic errors in Eq. (3), and the accuracy achievable on variation measurements depends only on the determination of the focal length of the contact.

IV. PINHOLE DIFFRACTION UNLIMITED

If we compare the accuracy obtained with the two pinhole system with the Rayleigh formula for the resolution power with a single circular opening, we see that the system of two pinholes is not diffraction limited. The resolution ρ according to the Rayleigh formula is

$$\rho = 1.22\lambda / d_p \sim 1 \text{ arcsec} \times (120 \text{ mm} / d_p). \quad (4)$$

Equation (4) gives a limit of $\rho = 30$ arcsec for a 4-mm pinhole, while $\rho = 10$ arcsec is the accuracy for the system of two pinholes with the same diameters.

The Rayleigh formula for the resolution power of a telescope describes, for a point-like source, the angular distance between the central peak and the first diffraction disk on the focal plane. If a second point-like source of equal intensity is located in the first dark disk, its central peak can be clearly identified by the naked eye under good atmospheric conditions. If two sources are closer than this angle, the only way to distinguish them as separate sources is to sample their central peaks with a large number N of photons. A long integration time t of photons in the detector is required

($N \propto t$). A Poisson uncertainty n proportional to \sqrt{N} is associated with the number of photons N that are the signal. This statistical uncertainty is called the noise n of the source. The noise is defined as the statistical variance of the signal $s = N$. The signal-to-noise ratio $s/n \propto \sqrt{N} \propto \sqrt{t}$ is the contrast between the source and its noise.⁵

For the Sun there is no problem of poor contrast (low signal-to-noise ratio). Note that the darkening of the solar limb, due to self-absorption of the solar atmosphere, does not affect the perception of the contact between the two images, because the solar limb is much brighter than the sky around it (high signal-to-noise ratio). The determination of the contact between the two images can be very accurate even with the naked eye, and this experiment gives results better than the classical Rayleigh limit calculated for a single pinhole. The abundance of photons determines the success of this experiment beyond the diffraction limit. If we did the same experiment with the Moon, the uncertainty would be much larger, because there are fewer photons, and the contrast would be much lower.

V. CONCLUSION

The idea of bringing into contact two opposite limbs of the Sun to measure its diameter was introduced by Savery and Bouguer in 1743–1748 with the invention of the heliometer.⁶ The latter is a telescope with a movable split-lens objective, and is employed for delicate measurements of the distance and relative direction of two stars too far apart to be easily measured in the field of view of an ordinary telescope (for example, the measurement of the parallax of 61 Cygni by Bessel in 1838).⁷

The Solar Disk Sextant (SDS) can be considered a modern heliometer. It is a balloon-born telescope for measuring the variations of the solar diameter with an expected accuracy of a few milliseconds of arc, where the two images at the focal plane are obtained with a glass wedge in front of the objective. Similar to the case discussed in Sec. IV, the performance of the SDS goes beyond the diffraction limit of the instrument. The principles of this experiment are explained in Refs. 8 and 9. Using a prism in front of the objective, the modern SDS telescope exploits the same principle of the heliometer developed in the Göttingen Observatory at the end of nineteenth century.¹⁰ In this respect our two-pinhole device is a good introduction to these more sophisticated astronomical instruments and techniques and represents their simpler prototype.

The two pinhole system gives an accuracy of 10 arcsec, which is 0.5% of the angular diameter. This accuracy is enough to detect the variation of 64 arcsec from the perihelion (January 4, Sun larger: 1952 arcsec) to aphelion (July 4, Sun smaller: 1888 arcsec) due to the orbital motion of the Earth.

The experiment is very simple to perform and allows students to actively learn about geometrical and wave optics, data reduction, signal-to-noise ratio and statistical uncertainty, the eccentricity of the Earth's orbit, the angular solar diameter and seasonal and intrinsic variations, and aspects of the history of classical astronomy.

It is instructive to compare the values obtained by this experiment with the values calculated by computer programs that reproduce the orbital parameters of the Earth and other astronomical objects. In particular, the program, X-EPHEM, is recommended.¹¹

ACKNOWLEDGMENTS

I am grateful to Dr. Orazio Converso and Professor Rosa Satullo who first supported me in presenting this project to high school students. I am also thankful to the director and the students of ITC Charles Darwin of Rome, and to Dr. Paul A. Sigismondi, a teacher at the Trinity School in New York City, and his students where this project was realized. Thanks also to Dr. Wayne H. Warren Jr., Dr. David W. Dunham, and their families, with whom I was a guest when I had this idea.

⁵Electronic mail: sigismondi@icra.it

¹Tycho Brahe, *Opera Omnia*, edited by J. L. E. Dreyer (Hauniae, 1913), Vol. XII, pp. 108–118.

²Johannes Kepler, *Ad Vitellionem Paralipomena, Quibus Astronomiae Pars Optica Traditur* (Frankfurt, 1604), Chap. IX.

³C. Sigismondi and F. Frascchetti, "Measurements of the solar diameter at Kepler's time," *The Observatory* **121**, 380–385 (2001).

⁴Kepler can be considered to be the inventor of the camera obscura and was the first to describe its principles (see Ref. 2). A camera obscura is a dark room where a pinhole is open on one wall. It projects an upside-down image of what is visible through the pinhole on the opposite wall. The formation of the image in a photographic camera is based on the same principle.

⁵P. R. Bevington and D. K. Robinson, *Data Reduction and Error Analysis for the Physical Sciences* (McGraw-Hill, New York, 1992).

⁶The Encyclopedia Britannica, 1911. See <http://www.1911ency.org/>. The original idea of the heliometer is attributed to the Danish astronomer Olaus Römer (1644–1710), although the date of the invention, 1768, given in the Web site of the scanned version of the 1911 Britannica is accidentally incorrect.

⁷See <http://scienceworld.wolfram.com/biography/Bessel.html>

⁸H. C. Chiu, E. Maier, K. H. Schatten, and S. Sofia, "Solar disk sextant optical configuration," *Appl. Opt.* **23**, 1230–1234 (1984).

⁹S. Sofia, H. C. Chiu, E. Maier, K. H. Schatten, P. Minott, and A. S. Endal, "Solar disk sextant," *Appl. Opt.* **23**, 1235–1237 (1984).

¹⁰W. Schur and L. Ambronn, "Die Messungen des Sonnendurchmessers," *Astron. Mitteilungen der Königlichen Sternwarte zu Göttingen* **7**, 17–60 (1905).

¹¹See <http://www.clearskyinstitute.com/xephem/> for information about XEPHEM.