The Cosmological Singularity

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Furthermore, if a singularity arises under general conditions, what is the analytical structure of this generic solution ?

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The BKL work showed that a singularity is a general property of a generic cosmological solution of the classical gravitational equations and not a consequence of the special symmetric structure of the exact models.

Most importantly, BKL were able to find the analytical structure of this generic solution and showed that its behaviour is of an extremely complex oscillatory character, of chaotic type.

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In spite of the complexity of the Einstein-matter field equations, the asymptotic behaviour of the fields near a cosmological singularity can be phrazed in surprisingly elementary terms involving finite-dimensional dynamical systems.

This description is valid generically, i.e. without making any symmetry assumption.

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These remarkable properties remain valid for string-related models.

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- explain the connection with fascinating mathematical structures ;
- briefly mention some open questions.

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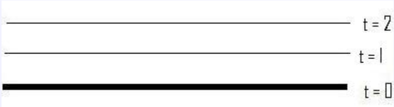

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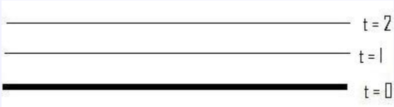

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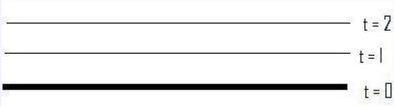

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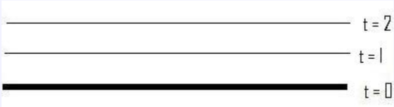

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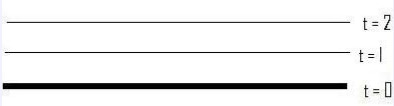

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This is remarkable given that the Einstein equations are non linear and remain so in the BKL limit.

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$$ds^2 = -dt^2 + a^2(t,\mathbf{x})\mathbf{l}^2 + b^2(t,\mathbf{x})\mathbf{m}^2 + c^2(t,\mathbf{x})\mathbf{n}^2$$

where a, b, c are the “scale factors" and $\mathbf{l} = l_i(t,\mathbf{x})dx^i$ etc

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- the non-diagonal components freeze so that the interesting dynamics is carried by scale factors ;
- the dynamics of the scale factors decouples at each spatial point (equations become in the limit ODE's with respect to time, a finite number at each point) - “ultralocality” ; not an assumption ! (“dimensional reduction without dimensional reduction”)

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- Terms due to the off-diagonal components also have a similar effect.
- This leads to a billiard picture, where the Kasner regime corresponds to free flight motion, and where the effect of the spatial gradients and of the off-diagonal terms is to introduce "walls" against which there are collisions, changing one Kasner regime to another in a never-ending way.

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There are two types of walls, “curvature walls” and “symmetry walls”.

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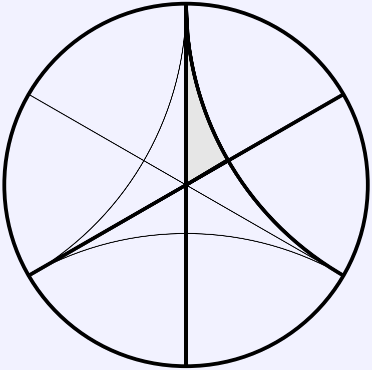

The billiard table for four-dimensional pure gravity (one billiard system at each point in space) in the Poincaré disk model. Straight lines are symmetry walls. Arc of circles are gravitationnal walls. Thick lines indicate the dominant walls. The billiard table, which lies on the positive side of each wall, is indicated in grey. It forms a triangle with angles $0, \frac{\pi}{3}, \frac{\pi}{2}$.

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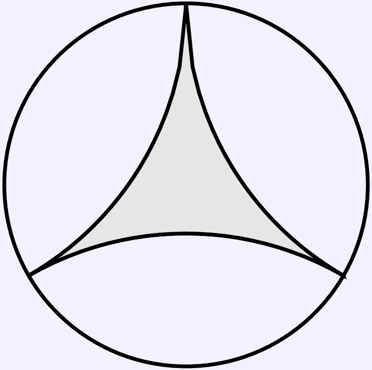

The billiard table for diagonal homogeneous cosmological models of Bianchi type IX. It is an ideal equilateral triangle formed by the three curvature (= gravitational) walls. The symmetry walls are absent. Any two different walls are parallel since they meet at infinity. The angles of this triangle are all equal to zero.

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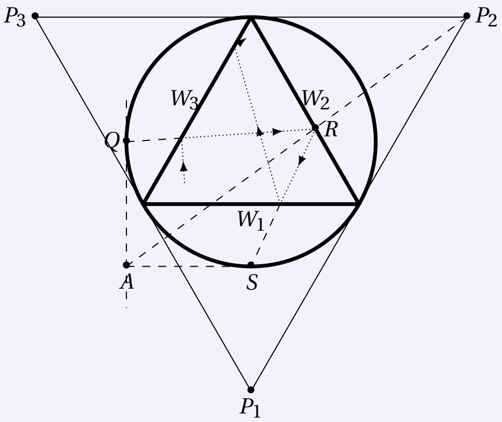

The motion is a succesion of reflections in the walls W_1, W_2 and W_3 and defines therefore an element in the group generated by the three reflections r_1, r_2, r_3 in the three walls of the billiard table (drawn in the Klein model representation).

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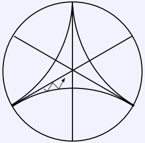

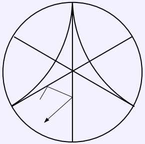

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- Dimension and shape of billiard table depends thus on spacetime dimension, menu of p-forms and dilaton couplings.
- Dynamics is chaotic if volume of billiard table is finite.

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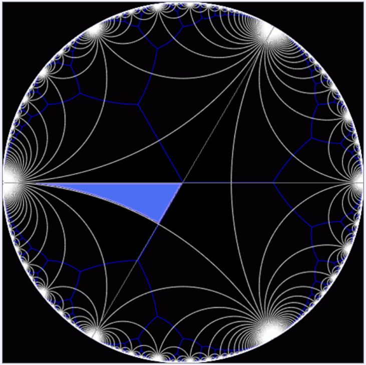
  where H_s^+ is the positive half-space bounded by the hyperplane H_s, and N is the number of relevant bounding hyperplanes.
- There are four types of walls defining the billiard region : symmetry walls, gravitational walls, p-form electric walls, p-form magnetic walls.

Angles between billiard walls



Angles between billiard walls : $0 = \frac{\pi}{\infty}, \frac{\pi}{3}, \frac{\pi}{2}$.

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so that the product $s_i s_j$ of the reflections s_i and s_j, which is is a rotation by the angle $\frac{2\pi}{m_{ij}}$, fulfills

Coxeter Polyhedra

For all string-related theories, the billiard table possesses the following remarkable properties :

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Coxeter Polyhedra

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Because of these features,

- the billiard group is a Coxeter group.
- The billiard region is a fundamental domain for the action of the billiard group (= group generated by the reflections against the walls).

Coxeter Polyhedra

Coxeter group for pure gravity in $D = 4$

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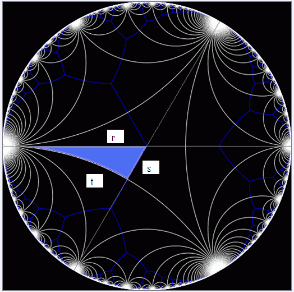

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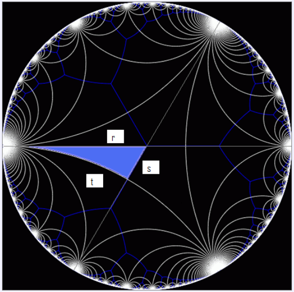

The reflections r, s and t, subject to the Coxeter relations

$$s^2 = 1, \ r^2 = 1, \ t^2 = 1, \ (rs)^3 = (sr)^3 = 1, \ (ts)^2 = (st)^2 = 1,$$

generate the symmetry group (which is a Coxeter group). There are no other relations (standard Coxeter presentation) and the region in blue is a fundamental domain.

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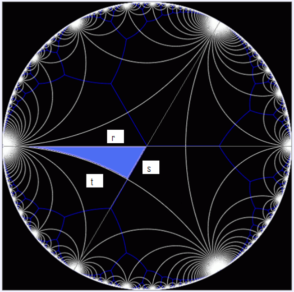

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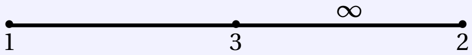
This is the group $PGL(2, \mathbb{Z})$ (also noted A_1^{++})

Coxeter graphs

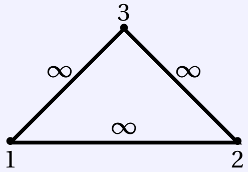
One can associate a so-called “Coxeter graph" to Coxeter reflection groups. The Coxeter graph encodes the relations $(s_i s_j)^{m_{ij}} = e$ among the generating reflections. If there is no relation among s_i and s_j, one conventionally declares $m_{ij} = \infty$. The graph contains as many vertices as generating reflections. For each pair of vertices, one draws an edge except when $m_{ij} = 2$, which corresponds to commuting reflections, $s_i s_j = s_j s_i$. Over each edge, one writes explicitly m_{ij}, except when $m_{ij} = 3$ which is the “default value".

Coxeter graphs

In the case of the two reflection groups $PGL(2,\mathbb{Z})$ and $G\Gamma(2)$ (group of the ideal triangle), there are three vertices and the Coxeter graphs read respectively :



The Coxeter graph for $PGL(2,\mathbb{Z})$

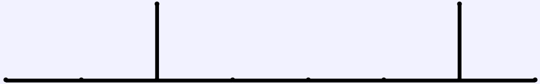


The Coxeter graph for $G\Gamma(2)$

Other theories

Hyperbolic simplex Coxeter groups are very exceptional structures. The maximum possible rank is 10, and there are only 3 hyperbolic simplex Coxeter groups of rank 10 ...

*

Name	Coxeter graph
$B_8^{++} \equiv BE_{10}$	
$D_8^{++} \equiv DE_{10}$	
$E_8^{++} \equiv E_{10}$	

... and these are all realized by string-related supergravities !

Crystallographic Coxeter groups

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A Coxeter group is crystallographic if and only if two conditions are satisfied : (i) The integers m_{ij} $(i \neq j)$ are restricted to be in the set $\{2, 3, 4, 6, \infty\}$, and (ii) for any closed circuit in the corresponding Coxeter graph, the number of edges labelled 4 or 6 is even.

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The Coxeter groups associated with string-related models are all simplex crystallographic Coxeter groups.

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Kac-Moody algebras are infinite-dimensional Lie algebras that generalize familiar finite-dimensional simple Lie algebras. The infinite-dimensional generalization preserves the key property of possessing a triangular decomposition such that the finite-dimensional concepts of Cartan subalgebra, raising and lowering operators, roots, positive roots, Weyl group, etc can all be introduced.

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Does this signal a hidden, infinite-dimensional symmetry that would exist independently of the cosmological context (the BKL analysis acting as a revelator) ?

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- Quantum effects

Book

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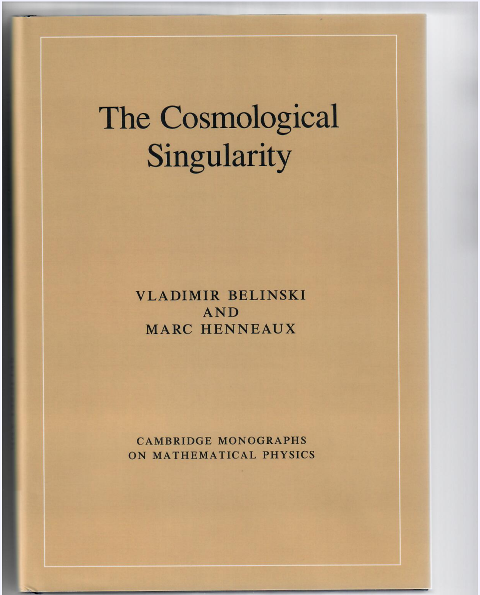
The Cosmological Singularity
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AND
MARC HENNEAUX
CAMBRIDGE MONOGRAPHS
ON MATHEMATICAL PHYSICS

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