DRIFT OF LIGHT RAYS INDUCED BY 
NONSYMMETRIC COSMIC FLOW:
an observational test of homogeneity of the Universe 
+ a few general comments on inhomogeneous models

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Contents

Part I (Nonsymmetric cosmic flow):
1 The Szekeres solutions 2
2 The redshift equations in the Szekeres models 4
3 Repeatable light paths 7
4 Examples of non-RLPs in the L–T model 12

Part II (Comments on inhomogeneous models) 18
5 Real-time cosmology 18
6 Erroneous ideas and paradigms 21

Part III (Appendices) 29
A Equations of general null geodesics in a Szekeres spacetime 29
B The propagation equations for $(\tau, \zeta, \psi)$ along a null geodesics in a Szekeres spacetime 32
C RLPs in shearfree normal models 36
D Dependence of RLPs on the observer congruence 42
Part I:

1 The Szekeres solution

The (quasi-spherical) Szekeres solution [1] used in this talk is defined by

\[ ds^2 = dt^2 - \frac{E^2(\Phi/E)_r^2}{1 - k(r)} dr^2 - \frac{\Phi^2}{E^2} (dx^2 + dy^2), \]

\[ E \equiv \frac{(x - P)^2}{2S} + \frac{(y - Q)^2}{2S} + \frac{S}{2}, \]

(1.1)

the \( k(r), M(r), P(r), Q(r) \) and \( S(r) \) are arbitrary functions; \( \Phi(t, r) \) obeys

\[ \Phi_{,t}^2 = -k(r) + \frac{2M(r)}{\Phi} + \frac{1}{3}\Lambda\Phi^2; \]

(1.2)

The mass density in energy units is

\[ \kappa \rho = \frac{2 (M/E^3)_{,r}}{(\Phi/E)^2 (\Phi/E)_{,r}}. \]

(1.3)

Eq. (1.2) implies that the bang time is in general position-dependent:

\[ \int_0^\Phi \frac{d\tilde{\Phi}}{\sqrt{-k + 2M/\tilde{\Phi} + \frac{1}{3}\Lambda\tilde{\Phi}^2}} = t - t_B(r). \]

(1.4)
\[ \text{ds}^2 = dt^2 - \frac{\mathcal{E}^2(\Phi/\mathcal{E})_r^2}{1 - k(r)} dr^2 - \frac{\Phi^2}{\mathcal{E}^2} (dx^2 + dy^2), \]

\[ \mathcal{E} = \frac{(x - P)^2}{2S} + \frac{(y - Q)^2}{2S} + \frac{S}{2}, \]

\[ \Phi,^2 = -k(r) + \frac{2M(r)}{\Phi} + \frac{1}{3}\Lambda \Phi^2. \]


The Friedmann limit follows when, in addition to the above, \(\Phi(t, r) = r R(t), k = k_0 r^2\) where \(k_0 = \text{const}\) and \(t_B\) is constant.
2 The redshift equations in the Szekeres models

Consider two light signals, the second one following the first after a short time-interval \( \tau \), both emitted by the same source and arriving at the same observer. The trajectory of the first signal is given by

\[
(t, x, y) = (T(r), X(r), Y(r)),
\]

the corresponding equation for the second signal is

\[
(t, x, y) = (T(r) + \tau(r), X(r) + \zeta(r), Y(r) + \psi(r)).
\]

While the first ray intersects a hypersurface \( r = r_0 \) at \( (t, x, y) = (T, X, Y) \), the second ray intersects the same hypersurface not only later, but, in general, at a different comoving location.

\( \Rightarrow \) In general the two rays will intersect different sequences of intermediate matter worldlines.

The same is true for nonradial rays in the L-T model.
This means that the second ray is emitted in a different direction and is received from a different direction by the observer.

Thus, a typical observer in a Szekeres spacetime should see each light source slowly *drift across the sky*. How slowly will be estimated further on.

We assume that \( (\zeta, \psi) \) and \( (d/d\tau)(\tau, \zeta, \psi) \) are small of the same order as \( \tau \), so we neglect all terms nonlinear in any of them and terms involving their products.
The symbol $\Delta f$ denotes

$$f(t + \tau, r, x + \zeta, y + \psi) - f(t, r, x, y)$$

*linearized in* $(\tau, \zeta, \psi)$.

Applying $\Delta$ to the null geodesic equations parametrised by $r$ we obtain the equations of propagation of $(\tau, \zeta, \psi)$ and $(\xi, \eta) \overset{\text{def}}{=} (d/dr)(\zeta, \psi)$ along a null geodesic (very complicated).
3 Repeatable light paths

There will be no drift when, for a given source and a given observer, each light ray, no matter when emitted, will proceed through the same intermediate sequence of matter world lines.

This happens only in some special situations (see further). Null geodesics having this property will be called repeatable light paths (RLP).

For a RLP we have

\[ \zeta = \psi = \xi = \eta = 0 \]  

(3.1)

all along the ray.
The equations of propagation become then:

\[
2 \left( \frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}^2}{\Phi^2} \right) \frac{dt}{dr} \frac{dx}{dr} \tau + 2 \frac{\Phi_{,t}}{\Phi} \frac{dx}{dr} \frac{d\tau}{dr}
- \frac{\Delta \Phi_1 E_{12}}{(1-k)\Phi} \frac{\Phi_{,t}}{\Phi} \frac{\Phi_{,t} E_{12}}{(1-k)\Phi^2} \\
+ 2 \left( \frac{\Delta \Phi_1}{\Phi} - \frac{\Phi_{,t} \tau}{\Phi^2} \right) \frac{dx}{dr} + \Delta U \frac{dx}{dr} = 0, 
\quad (3.2)
\]

\[
2 \left( \frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}^2}{\Phi^2} \right) \frac{dt}{dr} \frac{dy}{dr} \tau + 2 \frac{\Phi_{,t}}{\Phi} \frac{dy}{dr} \frac{d\tau}{dr}
- \frac{\Delta \Phi_1 E_{13}}{(1-k)\Phi} \frac{\Phi_{,t}}{\Phi} \frac{\Phi_{,t} E_{13}}{(1-k)\Phi^2} \\
+ 2 \left( \frac{\Delta \Phi_1}{\Phi} - \frac{\Phi_{,t} \tau}{\Phi^2} \right) \frac{dy}{dr} + \Delta U \frac{dy}{dr} = 0, 
\quad (3.3)
\]
The meaning of the symbols is

\[
\begin{align*}
\Phi_{,r} - \Phi \mathcal{E}_{,r} / \mathcal{E} & \overset{\text{def}}{=} \Phi_1, \\
\Phi_{,tr} - \Phi_{,t} \mathcal{E}_{,r} / \mathcal{E} & \overset{\text{def}}{=} \Phi_{01}, \\
\Phi_{,rr} - \Phi \mathcal{E}_{,rr} / \mathcal{E} & \overset{\text{def}}{=} \Phi_{11}, \\
\mathcal{E}_{,r} \mathcal{E}_{,x} - \mathcal{E} \mathcal{E}_{,xr} & \overset{\text{def}}{=} E_{12}, \\
\mathcal{E}_{,r} \mathcal{E}_{,y} - \mathcal{E} \mathcal{E}_{,yr} & \overset{\text{def}}{=} E_{13},
\end{align*}
\]

\[
\begin{align*}
\frac{d^2 r}{ds^2} & = \left( \frac{dr}{ds} \right)^2 \left\{ -2 \frac{\Phi_{01}}{\Phi_1} \frac{dt}{dr} - \left( \frac{\Phi_{11}}{\Phi_1} - \frac{\mathcal{E}_{,r}}{\mathcal{E}} + \frac{1}{2} k_{,r} \right) \\
& \quad - 2 \frac{\Phi}{\mathcal{E}^2} \frac{E_{12}}{\Phi_1} \frac{dx}{dr} - 2 \frac{\Phi}{\mathcal{E}^2} \frac{E_{13}}{\Phi_1} \frac{dy}{dr} + \frac{\Phi}{\mathcal{E}^2} \frac{1 - k}{\Phi_1} \Sigma \right\} \\
& \overset{\text{def}}{=} U(t, r, x, y) \left( \frac{dr}{ds} \right)^2.
\end{align*}
\]
These equations can be used in 2 ways:

1. As the condition (on the metric) for all null geodesics to be RLPs.

2. As the conditions under which special null geodesics are RLPs in subcases of the Szekeres spacetime.

In the first interpretation, (3.2) – (3.3) should be identities in the components of \( dx^\alpha / dr \), and this happens when

\[
\Psi \overset{\text{def}}{=} \Phi_{,tr} - \Phi_{,t} \Phi_{,r} / \Phi = 0.
\]  

(3.10)

This means zero shear, i.e. the Friedmann limit.

Thus, we have proven the following:

**Corollary:**

The only spacetimes in the Szekeres family in which all null geodesics have repeatable paths are the Friedmann models.
In the second interpretation of (3.2) – (3.3), there are only 2 nontrivial (i.e. non-Friedmannian) cases:

A. When the Szekeres spacetime is axially symmetric ($P$ and $Q$ are constant).

In this case, the RLPs are those null geodesics that stay on the axis of symmetry in each 3-space of constant $t$.

B. When the Szekeres spacetime is spherically symmetric (($P, Q, S$) are all constant) – then it reduces to the L–T model.

In this case, the radial null geodesics are the only RLPs that exist. All other null geodesics are not RLPs.

A formal proof of this statement is highly complicated, see [4].
4 Examples of non-RLPs in the L–T model

In Example 1 we use Profile 1, the observer and the light source both at 3.5 Gpc from the center of the void, the directions to them at the angle 1.8 rad. The other data are [5]

\[ t_B = 0, \]
\[ \rho(t_0, r) = \rho_0 \left[ 1 + \delta - \delta \exp \left( -\frac{r^2}{\sigma^2} \right) \right], \quad r \overset{\text{def}}{=} R(t_0, r). \]
\[ \rho_0 \overset{\text{def}}{=} \rho(t_0, 0) = 0.3 \times \left( 3H_0^2 \right)/\left( 8\pi G \right) \equiv 0.3\rho_{\text{critical}} \]
\[ H_0 = 72 \text{ km s}^{-1} \text{ Mpc}^{-1}, \]
\[ \delta = 4.05, \quad \sigma = 2.96 \text{ Gpc}. \]
Figure 1: Example 1

Nonradial null geodesics projected on the space $t = \text{now}$ along the flow lines of the L−T dust.

**Middle line: the ray received at the current instant.**

**Upper line: the ray received $5 \times 10^9$ years ago.**

**Lower line: the ray to be received $5 \times 10^9$ years in the future.**
The *time-averaged* rate of change of the position of the source in the sky, seen by the observer is

\[
\frac{\text{angle between the initial and final direction}}{\text{time interval}} = \frac{5 \times 10^9 \text{ years}}{\sim 10^{-7} \frac{\text{arcsec}}{\text{year}}}. \quad (4.1)
\]

The rate of drift in the next figures is calculated in the same way.
The observer $O$ is at $R_0$ from the center; the angle between the direction toward the galaxy ($\star$) and toward the origin is $\gamma$.

With this configuration we study 3 other examples. All have $d = 1$ Gyr $\approx 306.6$ Mpc, but different $R_0$, and one has Profile 2. For each $\gamma$ we calculate the rate of change $\dot{\gamma}$ by eq. (4.1).
Example 2: $R_0 = 3$ Gpc, Profile 1;
Example 3: $R_0 = 1$ Gpc, Profile 1;
Example 4: $R_0 = 1$ Gpc, Profile 2 (a deeper void in a higher-density background).
Figure 4:
\( \dot{\gamma} \) as a function of direction, in arcsec/(year \( \times 10^7 \)).

**Solid line:** case (2),
**dashed line:** case (3),
**dotted line:** case (4).

The amplitude is \( \sim 10^{-7} \) for (3) and \( \sim 10^{-6} \) for (2) and (4).
With the Gaia accuracy\(^1\) of \( 5 - 20 \times 10^{-6} \) arcsec, we would need a few years to detect this effect.

\(^1\)http://sci.esa.int/science-e/www/area/index.cfm?fareaid=26
Part II

5 Real-time cosmology

The non-RLP phenomenon was predicted by a different method (and under the name of *cosmic parallax*), by Claudia Quercellini, Luca Amendola *et al.* [6, 7].

Note what observing this drift would amount to: *we would be seeing, in real time, the Universe expand.*

There are more ways in which the expansion of the Universe might be directly observed. The Italian team composed them into a new paradigm, which they termed *real-time cosmology.*

One of them is the *redshift drift:* the change of redshift with time for any selected light source, induced by the expansion of the Universe.
Consider, as an example, an L–T model with $\Lambda$:

$$ds^2 = dt^2 - \frac{R_{tr}^2}{1 + 2E(r)} dr^2 - R^2(t, r) \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right), \quad (5.1)$$

$$R_{,t}^2 = 2E(r) + \frac{2M(r)}{R} + \frac{1}{3} \Lambda R^2. \quad (5.2)$$

Along a single radial null geodesic, directed \textit{toward} the observer, $t = T(t_o, r)$ (where $t_o$ is the instant of observation) the redshift is \cite{8}

$$1 + z(t_o, r) = \exp \left[ \int_{r_{\text{em}}}^{r_{\text{obs}}} \frac{R_{,t} \left( T(t_o, r), r \right)}{\sqrt{1 + 2E(r)}} dr \right]. \quad (5.3)$$

For any fixed source (i.e. constant $r$), eq. (5.2) defines a different expansion velocity $R_{,t}$ for $\Lambda = 0$ and for $\Lambda \neq 0$, and thus allows us to calculate the contribution of $\Lambda$ to $z$ via (5.3).

With the evolution type known, the expansion velocity depends on $r$, and so allows us to infer the distribution of mass along the past light cone.
According to the authors of [6, 7], the “European Extremely Large Telescope” (now in the planning, to be built in Chile) could detect the redshift drift during less than 10 years of constant monitoring a given light source.

The Gaia observatory could achieve this during about 30 years.

More on redshift drift was said in the contribution to the CM2 (dark energy) workshop by P. Mishra, M.-N. Celerier and T. Singh.
6 Erroneous ideas and paradigms

The L–T model was noticed in the astrophysics community – BUT:

1. Most astrophysicists treat it as an enemy to kill rather than as a useful new device. (Citation from Ref. [7]: Gaia or E-ELT could distinguish FLRW from L–T “possibly eliminating an exotic alternative explanation to dark energy”).

2. Astrophysicists practise (even insist on) a loose approach to mathematics. An example is to take for granted every equation found in any paper, without attention being paid to the assumptions under which it was derived.

Papers written in such a style planted errors in the literature, which then came to be taken as established facts.

In this section a few characteristic errors are presented (marked by •) together with their explanations (marked by *).
The accelerating expansion of the Universe is an observationally established fact (many refs., the Nobel Committee among them).

The established fact is the smaller than expected observed luminosity of the SNIa supernovae (but even this is obtained assuming that FLRW is the right cosmological model).

The accelerating expansion is an element of theoretical explanation of this observation.

When the SNIa observations are interpreted against the background of a suitably adjusted L–T model, they can be explained by matter inhomogeneities along the line of sight, with decelerating expansion [9, 10].
Fitting an L–T model to number counts or the $D_L(z)$ relation results in predicting a huge void, several hundred Mpc in radius, around the centre (too many papers to be cited, literature still growing). Measurements of the Fourier components of the CMB radiation then imply that our Galaxy should be very close to the center of this void, which contradicts the “cosmological principle”.

* The implied huge void is a consequence of hand-picked constraints imposed on the L–T model, for example constant $t_B$. When the model is employed at full generality, the giant void is not implied [11].

* The cosmological principle is a postulate, not a law of Nature, it cannot say which model is “right” and which is “wrong”.
• The L–T models used to explain away dark energy must have their bang-time function constant or else they “can be ruled out on the basis of the expected cosmic microwave background spectral distortion” [13].

* The papers that claim this stick to the “huge void” idea that sets them on the wrong track from the beginning.

* In order to meaningfully test any cosmological model against observations, one must apply it at every step of analysis of the observational data.
To do so, would require a re-analysis of a huge pool of data. C. Hellaby with coworkers [12] is working on such a program applied to the L–T model, but the work is far from being completed.

Lacking any better chance, we currently use observations interpreted in the FLRW framework to infer about the $M(r)$ and $t_B(r)$ functions in the L–T model.
This is justified as long as we intend to point out possibilities, under the tacit assumption that these results will be verified in the future within a complete revision of the observational material on the basis of the L–T model.

However, putting “precise” bounds on the L–T model functions using the self-inconsistent mixture of FLRW/L–T data available today is a self-delusion.

An example: the spatial distribution of galaxies and voids is inferred from the luminosity distance vs. redshift relation that applies only in the FLRW models. Without assuming the FLRW background, we know nothing about this distribution until we reconstruct it using the L–T model from the beginning.
The L–T and Szekeres models cannot be treated as exact models of the Universe, to be taken literally in all their aspects. They are *exact as solutions of Einstein’s equations*, but when applied in cosmology, they are merely *the next step of approximation after FLRW*.

If the FLRW approximation is good for some purposes, then a more detailed model, *when applied in a situation, in which its assumptions are fulfilled*, can only be better.
References


Part III:

A Equations of general null geodesics in a Szekeres spacetime

From the equations of null geodesics it is seen that a geodesic on which \( \frac{dr}{ds} = 0 \) over some open segment has \( \frac{dx}{ds} = \frac{dy}{ds} = 0 \) in that segment, and so is timelike. However, isolated points at which \( \frac{dr}{ds} = 0 \) can exist. Thus, \( r \) can be used as a (nonaffine) parameter on null geodesics on such segments where \( \frac{ds}{dr} > 0 \) or \( \frac{ds}{dr} < 0 \) throughout.
We introduce the following abbreviations:

\[ \Phi_r - \Phi \mathcal{E}_r / \mathcal{E} \stackrel{\text{def}}{=} \Phi_1, \quad \Phi_{tr} - \Phi_t \mathcal{E}_r / \mathcal{E} \stackrel{\text{def}}{=} \Phi_{01}, \quad \Phi_{rr} - \Phi \mathcal{E}_{rr} / \mathcal{E} \stackrel{\text{def}}{=} \Phi_{11}, \quad \mathcal{E}_r \mathcal{E}_x - \mathcal{E} \mathcal{E}_{xr} \stackrel{\text{def}}{=} E_{12}, \quad \mathcal{E}_r \mathcal{E}_y - \mathcal{E} \mathcal{E}_{yr} \stackrel{\text{def}}{=} E_{13}, \]

\[ \left( \frac{dx}{dr} \right)^2 + \left( \frac{dy}{dr} \right)^2 \stackrel{\text{def}}{=} \Sigma. \]

We have:

\[
\frac{d^2 r}{ds^2} = \left( \frac{dr}{ds} \right)^2 \left\{ -2 \frac{\Phi_{01}}{\Phi_1} \frac{dt}{dr} - \left( \frac{\Phi_{11}}{\Phi_1} - \frac{\mathcal{E}_r}{\mathcal{E}} + \frac{1}{2} \frac{k_r}{1 - k} \right) \right. \\
- \left. 2 \frac{\Phi}{\mathcal{E}^2} \frac{E_{12}}{\Phi_1} \frac{dx}{dr} - \frac{\Phi}{\mathcal{E}^2} \frac{E_{13}}{\Phi_1} \frac{dy}{dr} + \frac{\Phi}{\mathcal{E}^2} \frac{1 - k}{\Phi_1} \Sigma \right\} \\
\stackrel{\text{def}}{=} U(t, r, x, y) \left( \frac{dr}{ds} \right)^2,
\]
Then, the geodesic equations parametrised by $r$ become:

\[ \frac{d^2t}{dr^2} + \frac{\Phi_1 \Phi_{01}}{1 - k} + \frac{\Phi \Phi_{,t}}{\mathcal{E}^2} \Sigma + U \frac{dt}{dr} = 0, \tag{A.8} \]

\[ \frac{d^2x}{dr^2} + 2 \frac{\Phi_{,t}}{\Phi} \frac{dt}{dr} \frac{dx}{dr} - \frac{1}{\Phi} \frac{\Phi_1}{1 - k} E_{12} \]
\[ + \frac{2 \Phi_1}{\Phi} dx \frac{\mathcal{E},x}{\mathcal{E}} \left( \frac{dx}{dr} \right)^2 - 2 \frac{\mathcal{E},y}{\mathcal{E}} \frac{dx}{dr} \frac{dy}{dr} \]
\[ + \frac{\mathcal{E},x}{\mathcal{E}} \left( \frac{dy}{dr} \right)^2 + U \frac{dx}{dr} = 0, \tag{A.9} \]

\[ \frac{d^2y}{dr^2} + 2 \frac{\Phi_{,t}}{\Phi} \frac{dt}{dr} \frac{dy}{dr} - \frac{1}{\Phi} \frac{\Phi_1}{1 - k} E_{13} \]
\[ + \frac{2 \Phi_1}{\Phi} dy \frac{\mathcal{E},y}{\mathcal{E}} \left( \frac{dx}{dr} \right)^2 - 2 \frac{\mathcal{E},x}{\mathcal{E}} \frac{dx}{dr} \frac{dy}{dr} \]
\[ - \frac{\mathcal{E},y}{\mathcal{E}} \left( \frac{dy}{dr} \right)^2 + U \frac{dy}{dr} = 0. \tag{A.10} \]
The propagation equations for \((\tau, \zeta, \psi)\) along a null geodesics in a Szekeres spacetime

\[
\frac{d^2 \tau}{dr^2} + \frac{\Phi_{01} \Delta \Phi_1 + \Phi_1 \Delta \Phi_{01}}{1 - k} + \frac{\left(\Phi_{,t}^2 + \Phi \Phi_{,tt}\right)}{\Sigma} \Sigma \tau
\]

\[
- \frac{2 \Phi_{,t} \Delta \Sigma}{\Sigma} + \frac{\Phi_{,t} \Delta \Sigma}{\Sigma} + \Delta U \frac{dt}{dr} + U \frac{d\tau}{dr} = 0, \quad (B.1)
\]
\[
\frac{d^2 \zeta}{dr^2} + 2 \left( \frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}^2}{\Phi^2} \right) \frac{dt}{dr} \frac{dx}{dr} \frac{d\tau}{dr} + 2 \frac{\Phi_{,t} dx}{dr} \frac{d\tau}{dr} + 2 \frac{\Phi_{,t} dt}{dr} \xi - \frac{\Delta \Phi E_{12}}{(1 - k) \Phi} + \frac{\Phi_{,t} \Phi E_{12} \tau}{(1 - k) \Phi^2} - \frac{\Phi \Delta E_{12}}{(1 - k) \Phi} \\
+ 2 \left( \frac{\Delta \Phi}{\Phi} - \frac{\Phi \Phi_{,t} \tau}{\Phi^2} \right) \frac{dx}{dr} + 2 \frac{\Phi_1}{\Phi} \xi \\
- \left( \frac{dx}{dr} \right)^2 \left( \frac{\zeta}{S \Phi} - \frac{E_{,x} \Delta \Phi}{E^2} \right) - \frac{2 E_{,x} \xi dx}{E} \\
- 2 \frac{dx}{dr} \frac{dy}{dr} \left( \frac{\psi}{S \Phi} - \frac{E_{,y} \Delta \Phi}{E^2} \right) - \frac{2 E_{,y}}{E} \left( \frac{dy}{dr} \xi + \frac{dx}{dr} \eta \right) \\
+ \left( \frac{dy}{dr} \right)^2 \left( \frac{\zeta}{S \Phi} - \frac{E_{,x} \Delta \Phi}{E^2} \right) + \frac{2 E_{,x} \eta dy}{E} \\
+ \Delta U \frac{dx}{dr} + U \xi = 0, \quad (B.2)
\]
\[
\frac{d^2 \psi}{dr^2} + 2 \left( \frac{\Phi_{,tt}}{\Phi} - \frac{\Phi_{,t}^2}{\Phi^2} \right) \frac{dt}{dr} \frac{dy}{dr} + 2 \frac{\Phi_{,t}}{\Phi} \frac{dy}{dr} \frac{d\tau}{dr} + 2 \frac{\Phi_{,t}}{\Phi} \frac{dy}{dr} \frac{d\tau}{dr} \\
+ 2 \frac{\Phi_{,t}}{\Phi} \frac{dt}{dr} \eta - \frac{\Delta \Phi_1 E_{13}}{(1 - k)\Phi} + \frac{\Phi_{,t} \Phi_1 E_{13} \tau}{(1 - k)\Phi^2} - \frac{\Phi_1 \Delta E_{13}}{(1 - k)\Phi} \\
+ 2 \left( \frac{\Delta \Phi_1}{\Phi} - \frac{\Phi_1 \Phi_{,t} \tau}{\Phi^2} \right) \frac{dy}{dr} + 2 \frac{\Phi_1}{\Phi} \eta \\
+ \left( \frac{dx}{dr} \right)^2 \left( \frac{2 \psi}{S \mathcal{E}} - \frac{\mathcal{E}_{,y} \Delta \mathcal{E}}{\mathcal{E}^2} \right) + 2 \mathcal{E}_{,y} \xi \frac{dx}{dr} \\
- 2 \frac{dx \, dy}{dr \, dr} \left( \frac{\zeta}{S \mathcal{E}} - \frac{\mathcal{E}_{,x} \Delta \mathcal{E}}{\mathcal{E}^2} \right) - 2 \frac{\mathcal{E}_{,x}}{\mathcal{E}} \left( \frac{dy}{dr} \xi + \frac{dx}{dr} \eta \right) \\
- \left( \frac{dy}{dr} \right)^2 \left( \frac{2 \psi}{S \mathcal{E}} - \frac{\mathcal{E}_{,y} \Delta \mathcal{E}}{\mathcal{E}^2} \right) - 2 \frac{\mathcal{E}_{,y} \eta \, dy}{\mathcal{E}} \frac{dy}{dr} \\
+ \Delta U \frac{dy}{dr} + U \eta = 0, \tag{B.3}
\]
In addition, we have the first integral of the geodesic equations:

\[
\left( \frac{dt}{dr} \right)^2 = \frac{(\Phi_1)^2}{1-k} + \frac{\Phi^2}{\mathcal{E}^2} \left[ \left( \frac{dx}{dr} \right)^2 + \left( \frac{dy}{dr} \right)^2 \right], \quad (B.4)
\]

Applying \( \Delta \) to this we get the first-order relation

\[
\frac{d\tau}{dr} \frac{dt}{dr} = \frac{\Phi_1 \Delta \Phi_1}{1-k} + \left( \frac{\Phi \Phi, \tau}{\mathcal{E}^2} - \frac{\Phi^2 \Delta \mathcal{E}}{\mathcal{E}^3} \right) \left[ \left( \frac{dx}{dr} \right)^2 + \left( \frac{dy}{dr} \right)^2 \right]
\]

\[
+ \frac{\Phi^2}{\mathcal{E}^2} \left( \frac{dx}{dr} \xi + \frac{dy}{dr} \eta \right). \quad (B.5)
\]

Note: \( dt/dr < 0 \) for an incoming ray.
C RLPs in shearfree normal models

In the Szekeres models, the condition for all null geodesics to be RLPs was the vanishing of shear. This suggests that the cause of the non-RLP phenomenon might be shear in the cosmic flow.

To test this supposition, the existence of RLPs was next investigated in those cosmological models in which shear (and rotation) is zero [3] – the shearfree normal models found by Barnes [1].

They obey the Einstein equations with a perfect fluid source and contain, as the acceleration-free limit, the whole FLRW family.

There are four classes of them: the Petrov type D metrics that are spherically, plane and hyperbolically symmetric, and the conformally flat metric found earlier by Stephani [2].
In the Petrov type D case, the metric in comoving coordinates is
\[ ds^2 = \left( \frac{FV_t}{V} \right)^2 dt^2 - \frac{1}{V^2} \left( dx^2 + dy^2 + dz^2 \right), \quad \text{(C.1)} \]
where \( F(t) \) is an arbitrary function, related to the expansion scalar \( \theta \) by \( \theta = 3/F \). The Einstein equations reduce to the single equation:
\[ w,uu/w^2 = f(u), \quad \text{(C.2)} \]
where \( f(u) \) is an arbitrary function, while \( u \) and \( w \) are related to \((x, y, z)\) and to \( V(t, x, y, z) \) differently in each subfamily. We have
\[
(u, w) = \begin{cases} 
(r^2, V) & \text{with spherical symmetry;} \\
(z, V) & \text{with plane symmetry;} \\
(x/y, V/y) & \text{with hyperbolic symmetry.}
\end{cases}
\]
\[ r^2 \overset{\text{def}}{=} x^2 + y^2 + z^2. \quad \text{(C.3)} \]

The FLRW limit follows when \( f = 0 \) and \( V = R(t)g(x, y, z) \).
The conformally flat Stephani solution [2, 5] has the metric given by (C.1),

\[
    ds^2 = \left( \frac{FV_t}{V} \right)^2 dt^2 - \frac{1}{V^2} \left( dx^2 + dy^2 + dz^2 \right),
\]

the coordinates are comoving, and \( V(t, x, y, z) \) is given by

\[
    V = \frac{1}{R} \left\{ 1 + \frac{1}{4} k(t) \left[ (x - x_0(t))^2 + (y - y_0(t))^2 + (z - z_0(t))^2 \right] \right\}, \tag{C.4}
\]

where \((R, k, x_0, y_0, z_0)\) are arbitrary functions of \( t \). This a generalisation of the whole FLRW class, which results when \((k, x_0, y_0, z_0)\) are all constant. In general, (C.4) has no symmetry.
In the most general cases, generic null geodesics are not RLPs. Consequently, it is not shear that causes the non-RLP property.

The necessary (but not sufficient) condition for all null geodesics to be RLPs is zero conformal curvature. But the drift-free subcases are less general than the conformally flat limit in the corresponding class, and more general than FLRW.

In the general type D shearfree normal models, the only RLPs are radial null geodesics in the spherical case and their analogues in the other two cases.

In the most general Stephani spacetime, RLPs do not exist. In the axially symmetric subcase of the Stephani solution the RLPs are those geodesics that intersect the axis of symmetry in every space of constant time. In the spherically-, plane- and hyperbolically symmetric subcases, the RLPs are the radial geodesics.
The defining property of the drift-free models is that their time-dependence in the comoving coordinates can be factored out, and the cofactor metric is static.

*The FLRW models have the same property.*
For example, in the spherically symmetric type D case:

\[
\begin{align*}
    ds^2 &= \frac{1}{V^2} \left\{ \left[ (A_1 + A_2 r^2) (FS, t d t) \right]^2 - dr^2 \\
    &\quad - r^2 \left( d\vartheta^2 + r^2 \sin^2 \vartheta d\varphi^2 \right) \right\}, \tag{C.5}
\end{align*}
\]

the whole non-staticity is contained in \( V \):

\[
    V = B_1 + B_2 r^2 + (A_1 + A_2 r^2) S(t). \tag{C.6}
\]

The \((A_1, A_2, B_1, B_2)\) are arbitrary constants and \(S(t)\) is an arbitrary function.

This model is conformally flat, but is more general than FLRW because the pressure in it is spatially inhomogeneous. The FLRW limit follows when \(A_1 \neq 0\) and \(B_2 = (A_2/A_1)B_1\).
D Dependence of RLPs on the observer congruence

The RLPs are defined relative to the congruence of worldlines of the observers and light sources. So far, we considered observers and light sources attached to the particles of the cosmic medium, whose velocity field is defined by the spacetime geometry via the Einstein equations.

But we could as well consider other timelike congruences, or spacetimes in which no preferred timelike congruence exists, for example Minkowski.

It turns out that even in the Minkowski spacetime one can devise a timelike congruence that will display the non-RLP property.
Take the Minkowski metric in the spherical coordinates
\[ ds^2 = dt'^2 - dr'^2 - r'^2 \left( d\vartheta^2 + \sin^2 \vartheta d\varphi^2 \right), \] (D.1)
and carry out the following transformation on it:
\[ t' = (r - t)^2 + 1/(r + t)^2, \quad r' = (r - t)^2 - 1/(r + t)^2. \] (D.2)
The result is the metric
\[ ds^2 = \frac{1}{(r+t)^4} \{ 16u (dt^2 - dr^2) - (u^2 - 1)^2 (d\theta^2 + \sin^2 \theta d\phi^2) \}, \]
\[ u \overset{\text{def}}{=} r^2 - t^2. \quad (D.3) \]

Now we assume that the curves with the unit tangent vector field
\[ u^\alpha = \left[ (r + t)^2 / (4\sqrt{u}) \right] \delta^\alpha_0 \]
are world lines of test observers and test light sources.

Proceeding as before we conclude that, with respect to this congruence, generic null geodesics in the Minkowski spacetime have non-repeatable paths. (The exception are those rays that are radial in the coordinates of (D.3)). This is because the time-dependence of (D.3) cannot be factored out.

44
References


