Averaging via Cartan scalars

Petr Kašpar, Otakar Svítek

Charles University in Prague

5. 7. 2012
Outline

1. Introduction
2. Cartan scalars
3. Averaging Cartan scalars
Motivation: Cosmology

- Traditional approach

\[ E_{\mu\nu} \left( \langle g_{\alpha\beta} \rangle \right) = 8\pi \langle T_{\mu\nu} \rangle \]

\[ \langle T_{\mu\nu} \rangle = \rho u_\mu u_\nu + p (g_{\mu\nu} + u_\mu u_\nu) \]

\[ \langle g_{\mu\nu} \rangle = -dt^2 + a^2(t) \left[ d\chi^2 + \Sigma^2 d\Omega^2 \right] \]

- "Correct" approach

\[ \langle E_{\mu\nu} (g_{\alpha\beta}) \rangle = 8\pi \langle T_{\mu\nu} \rangle \]

\[ E_{\mu\nu}(\langle g_{\mu\nu} \rangle) = 8\pi \langle T_{\mu\nu} \rangle + E_{\mu\nu}(\langle g_{\mu\nu} \rangle) - \langle E_{\mu\nu}(g_{\mu\nu}) \rangle \]
Averaging methods

- Isaacson's approach
  \[ \langle A_{\mu\nu}(x) \rangle_{BH} = \frac{1}{V_D} \int_D g_{\mu}^{\alpha'}(x, x') g_{\nu}^{\beta'}(x, x') A_{\alpha' \beta'}(x') \sqrt{-g(x')} d^4x' \]

- Macroscopic Gravity
  \[ \tilde{t}_{\alpha\ldots}(x) = \frac{1}{V_D} \int_D \tilde{t}_{\alpha\ldots}(x, x') \sqrt{-g'} d^n x' \]
  \[ \tilde{t}_{\alpha\ldots}(x, x') = \mathcal{W}_{\alpha'}^{\alpha}(x', x) \ldots \mathcal{W}_{\beta'}^{\beta}(x', x) \ldots t_{\alpha'\ldots}(x') \]

Bilocal operators \( \mathcal{W}_{\beta}^{\alpha'} \) satisfy

\[ \lim_{x' \to x} \mathcal{W}_{\beta}^{\alpha'}(x', x) = \delta_{\beta}^{\alpha} \]

\[ \mathcal{W}_{\gamma''}^{\alpha'}(x', x'') \mathcal{W}_{\beta}^{\gamma''}(x'', x) = \mathcal{W}_{\beta}^{\alpha'}(x', x). \]
It can be shown that Riemann tensor and the finite number of its covariant derivatives in a moving frame (called Cartan scalars) completely (locally) specify the geometry of Riemannian manifold.

We will define all geometrical objects on the enlarged $\frac{1}{2}n(n+1)$ dimensional space - the bundle of frames $F(M)$. Cartan structure equations read

$$d\omega^i = \omega^j \wedge \omega^i_j$$

$$d\omega^i_j = -\omega^i_k \wedge \omega^k_j + \frac{1}{2}R^i_{jkl}\omega^k \wedge \omega^l$$

To generate covariant derivative of the Riemann tensor, we apply repeatedly exterior derivative.

$$dR_{ijkl} = R_{m jkl}\omega^m_i + R_{imkl}\omega^m_j + R_{ijml}\omega^m_k + R_{ijkm}\omega^m_l + R_{ijkl; m}\omega^m,$$

$$dR_{ijkl; n} = R_{mjkl; n}\omega^m_i + R_{imkl; n}\omega^m_j + ... + R_{ijkl; nm}\omega^m,$$
Let \( R^p \) denote the set \( \{ R_{ijkm}, R_{ijkm;n_1}, \ldots, R_{ijkm;n_1\ldots n_p} \} \) where \( p \) is such that \( R^{p+1} \) contains no element that is functionally independent of the elements in \( R^p \). Two functions are functionally independent if the one form \( df \) and \( dg \) are linearly independent. Then the set \( R^{p+1} \) is called Cartan scalars.

Here is the Cartan-Karlehedehede algorithm how to construct the set \( R^{p+1} \)

1. Let \( q = 0 \).
2. Compute \( R^q \) (\( R^0 \) is the empty set).
3. Find \( H_q \), the isotropy group of \( R^q \) (\( H_q \subseteq H_{q-1} \)), which leaves the tetrad components of the Riemann tensor unchanged.
4. Determine the frame (up to \( H_q \)) by requiring that \( R^q \) takes a standard form.
5. Find \( t_q \), the number of independent functions in \( R^q \).
6. If \( t_q = t_{q-1} \) and \( \dim H_q = \dim H_{q-1} \), then \( q = p + 1 \).
Otherwise, increase \( q \) by one and repeat from step 2.
Cartan scalars - constrains

- $\omega^i_j$ and $\omega^k$ are independent on $F(M)$ and we can denote them as $\{\omega^I\} \equiv \{\omega^i, \omega^j\}$, $I=1,2,\ldots \frac{1}{2}n(n+1)$. Cartan structure equations can be rewritten into the simple form

$$d\omega^I = \frac{1}{2} C^I_{JK} \omega^J \wedge \omega^K.$$

- We will choose a maximal set of functionally independent objects in $\mathbb{R}^p$ as $I^\alpha$, $\alpha=1,\ldots,k \leq \frac{1}{2}n(n+1)$.

$$dC^I_{JK} = C^I_{JK,\alpha} dI^\alpha \equiv C^I_{JK,\alpha} I^\alpha_{|L} \omega^L \equiv C^I_{JK|L} \omega^L,$$

$$dC^I_{JK|L} = C^I_{JK|LM} \omega^M,$$

...
Introduction
Cartan scalars
Averaging Cartan scalars

Cartan scalars - constrains

- There is the relation between the one-form $dI^\alpha$ and $\omega^L$.

$$dI^\alpha = I^\alpha |_L \omega^L.$$ 

- Necessary and sufficient conditions to construct the geometry are

$$d^2 I^\alpha = 0,$$

$$I^\alpha |_{K,\beta} I^\beta |_{J} - I^\alpha |_{J,\beta} I^\beta |_{K} + I^\alpha |_{L} C^L_{JK} = 0,$$

$$d^2 \omega^P = 0,$$

$$C^P_{[JK|L]} + C^P_{M[K} C^M_{LJ]} = 0.$$ 

where $P = k + 1, k + 2, ..., \frac{1}{2} n(n + 1)$. 

Petr Kašpar, Otakar Svítek
Averaging via Cartan scalars
Averaging Cartan scalars

- Averaging using scalar curvature invariants (Coley, 2010)
- Integration of the scalar functions $f \in R^{p+1}$ according to the rule

$$\bar{f}(x) = \frac{1}{V_\Omega} \int_{\Omega} f(x') d^N x'$$

- There are two goals of averaging - the first is an averaging of the spacetime geometry and the second is an averaging of the Einstein equations. Einstein tensor (rewritten in a tetrad form) consists of the finite sum of Cartan scalars, so we can obtain correction to the left hand side of the Einstein equation.
There are two approaches

- From a given metric compute Cartan scalars, choose minimal set, perform averaging and obtain averaged metric (not necessarily simple).
  
  Problem: When the frame is fixed, the set of Cartan scalars $R^{p+1}$ can be replaced by the set $\left\{ R^p_{qkl}, \gamma^a_{bi}, x^\alpha_i, \eta_{ij} \right\}$.

- How to average tetrad???

- Guess the form of the averaged metric and compare averaged Cartan scalars of a given metric and Cartan scalars of an averaged metric, which by the definition have to satisfy the constraints.
We will consider flat FRW metric
\[ ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2) \]

Nonzero Cartan scalars are
\[ \phi_{00}' = \phi_{22}' = 2\phi_{11}' = -\frac{1}{2}a^{-1}a_{tt} + \frac{1}{2}a^{-2}(a_t)^2, \]
\[ \Lambda = \frac{1}{4}a^{-1}a_{tt} + \frac{1}{4}a^{-2}(a_t)^2, \]
\[ D\phi_{00}' = D\phi_{33}' = 3D\phi_{11}' = 3D\phi_{22}' = -\frac{1}{2\sqrt{2}}a^{-1}a_{ttt} \]
\[ + \frac{5}{2\sqrt{2}}a^{-2}a_t a_{tt} - \sqrt{2}a^{-3}(a_t)^3, \]
\[ D\Lambda_{00}' = D\Lambda_{11}' = \frac{1}{4\sqrt{2}}a^{-1}a_{ttt} + \frac{1}{4\sqrt{2}}a^{-2}a_t a_{tt} - \frac{1}{2\sqrt{2}}a^{-3}(a_t)^3. \]
The Lemaître-Tolman-Bondi (LTB) metric is the spherical symmetric exact solution of the Einstein equations. It describes inhomogeneous dust with the stress energy tensor

$$T_{\mu\nu} = \rho u_\mu u_\nu$$

The line element reads

$$ds^2 = -dt^2 + \frac{(R')^2}{1 + 2E(r)} dr^2 + R^2(t, r)(d\theta^2 + \sin^2(\theta)d\phi^2)$$

Function $R(t, r)$ obeys the Einstein equation

$$R_{,t}^2 = 2E + \frac{2M}{R} + \frac{\Lambda}{3} R^2$$

The energy density $\rho$ is determined by the equation

$$4\pi \rho = \frac{M'}{R' R^2}$$
Averaging LTB spacetime

- We will assume the WKB form $R(t, r) = A(t, r) \exp \psi(t, r)$
- Nontrivial zero-order Cartan scalars are

$$\psi_2 = -\frac{1}{6} (R, r)^{-1} R_{ttr} + \frac{1}{6} R^{-1} R_t (R, r)^{-1} R_{tr} + \frac{1}{6} R^{-1} R_{tt} - \frac{1}{6} R^{-2} (R, t)^2,$$

$$\phi_{00'} = \phi_{22'} = \frac{1}{2} R^{-1} R_t (R, r)^{-1} R_{tr} - \frac{1}{2} R^{-1} R_{tt},$$

$$\phi_{11'} = -\frac{1}{4} (R, r)^{-1} R_{ttr} + \frac{1}{4} R^{-2} (R, t)^2,$$

$$\Lambda = \frac{1}{12} (R, r)^{-1} R_{ttr} + \frac{1}{6} R^{-1} R_t (R, r)^{-1} R_{tr} + \frac{1}{6} R^{-1} R_{tt} + \frac{1}{12} R^{-2} (R, t)^2.$$  

- If we plug the WKB form into the spinors, in the leading order all quantities are equal to zero except $\Lambda = \frac{1}{2} \psi^2_t$
In this approximation they are higher-order Cartan scalars (in the leading order) all equal to zero. For example the easiest one is

\[D\phi_{00'} = \frac{1}{2\sqrt{2}} R^{-1} R_{,t}(R,_{r})^{-1} R_{,ttr} - \frac{3}{2\sqrt{2}} R^{-1} R_{,t}(R,_{r})^{-2}(R,_{tr})^{2}\]

\[+ \frac{1}{2\sqrt{2}} R^{-1} R_{,t}(R,_{r})^{-2} R_{,ttr} - \frac{1}{2\sqrt{2}} R^{-1} R_{,t}(R,_{r})^{-3} R,_{tr} R,_{rr}\]

\[- \frac{1}{2\sqrt{2}} R^{-1} R,_{ttt} + \frac{3}{2\sqrt{2}} R^{-1}(R,_{r})^{-1} R,_{tt} R,_{tr} - \frac{1}{2\sqrt{2}} R^{-1}(R,_{r})^{-1} R,_{ttr}\]

\[+ \frac{1}{2\sqrt{2}} R^{-1}(R,_{r})^{-2}(R,_{tr})^{2} - \frac{1}{2\sqrt{2}} R^{-2}(R,_{t})^{2}(R,_{r})^{-1} R,_{tr}\]

\[+ \frac{1}{2\sqrt{2}} R^{-2} R,_{t} R,_{tt} - \frac{1}{2\sqrt{2}} R^{-2} R,_{t}(R,_{r})^{-1} R,_{tr} + \frac{1}{2\sqrt{2}} R^{-2} R,_{tt}\]
Thank You!