ON THE ANALOG GRAVITY FORMALISM APPLIED TO WHITE DWARFS.

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Structure of the talk

- Short introduction to Analog Gravity formalism.
- The classical white dwarf (WD) “à la Chandrasekhar”.
- A novel analytical treatment of this classical problem through a parametric Padè approximation (comparison with numerical results).
- General perturbations of the white dwarf problem via Clebsch formulation of hydrodynamics.
- Clebsch formulation re-interpreted through analog gravity.
- Comparison of the acoustic light-cone structure with the numerical integration of WD full perturbative equations.
- Addition of rotation: the n=1 polytropic William’s solution.
- Discussion.
What is analog gravity?

Some decades ago, it was shown by **W. Unruh** that in a **perfect fluid**, in ordinary Newtonian dynamics, linear irrotational perturbations behave as a **massless scalar field in General Relativity**.

It emerges naturally from the **hydrodynamical** equations then a **four dimensional Lorenzian curved manifold** similar to the ones of Einstein’s Theory, with the **speed of sound** replacing the **speed of light**.
What is analog gravity?

Why is this analogy useful?

1. in order to find condensed matter lab counterparts of General Relativistic effects (acoustic black holes, Hawking effect, ...)

2. in order to see if non relativistic problems can be revisited through General Relativity concepts (geodesics, light cones, quasi-normal modes, event horizons, ...)

A popular complete review

LIVING REVIEWS in relativity
Vol. 8 (2005) > lrr-2005-12
Living Rev. Relativity 8 (2005), 12
http://www.livingreviews.org/lrr-2005-12

Analogue Gravity
Carlos Barceló and Stefano Liberati and Matt Visser
Published Works

PHYSICAL REVIEW D 83, 064039 (2011)

Effective geometry of a white dwarf

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(Received 4 November 2010; published 29 March 2011)

Results: analog gravity applied to self-gravitating quantum systems; not Irrotational.
Published Works

PHYSICAL REVIEW D 83, 064039 (2011)

Effective geometry of a white dwarf

PHYSICAL REVIEW D 82, 044005 (2010)

Effective geometry of the $n = 1$ uniformly rotating self-gravitating polytrope

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(Received 22 February 2010; published 5 August 2010)

Results: analog gravity applied to self-gravitating polytropic systems but not irrotational anymore.
Published Works

PHYSICAL REVIEW D 83, 064039 (2011)

Effective geometry of a white dwarf

PHYSICAL REVIEW D 82, 044005 (2010)

Effective geometry of the $n = 1$ uniformly rotating self-gravitating polytropic system. Irrotationality.

PHYSICAL REVIEW D 78, 064024 (2008)

Effective geometries in self-gravitating polytropes

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The White Dwarf

i.e. a totally degenerate self-gravitating quantum Fermi electron gas.

Equation of equilibrium

\[ \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho. \]

\[ \frac{1}{\eta^2} \frac{d}{d\eta} \left( \eta^2 \frac{d\phi}{d\eta} \right) = -\left( \phi^2 - \frac{1}{y_0^2} \right)^{3/2} \]

\( \phi(0) = 1 \) and \( d\phi/d\eta \big|_{\eta=0} = 0. \)

Solved in the range \( \eta = [0, \eta_1]. \)

\( \eta_1 \) = stellar surface

\( \phi(\eta_1) = 1/y_0 \)

We follow Chandrasekhar’s book

\( P = Af(x), \quad \rho = Bx^3, \quad x = x(r) \)

\( f(x) = x(2x^2 - 3)(x^2 + 1)^{1/2} + 3\sinh^{-1}x. \)

\( A = 6.01 \times 10^{22}, \quad B = 9.82 \times 10^5 \mu_e \)

\( \mu_e \) = Electron molecular weight

\[ \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\sqrt{x^2 + 1}}{dr} \right) = -\frac{\pi GB^2}{2A} x^3. \]

\( y^2 = x^2 + 1 \)

\( \alpha = 1/(By_0)\sqrt{2A/(\pi G)} \)

\( r = \alpha \eta \) and \( y = y_0 \phi \)

\( \frac{\gamma^2}{\gamma_0^2} = x^2 + 1 \)

\( x_0 \) is the value of \( x \) at the center
Numerical Integration

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<th>$1/y_0^2$</th>
<th>$\eta_1$ (numerical)</th>
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Taylor Series (up to power 32)

$$\phi = 1 - \frac{q^3}{6} \eta^2 + \frac{q^4}{40} \eta^4 - \frac{q^5 (5q^2 + 14)}{7!} \eta^6$$

$$+ \frac{q^6 (339q^2 + 280)}{3 \times 9!} \eta^8$$

$$- \frac{q^7 (1425q^4 + 11436q^2 + 4256)}{5 \times 11!} \eta^{10} + \ldots$$

$$q^2 = (y_0^2 - 1)/y_0^2.$$
The approximate stellar radius can be solved easily by using standard algebra formulas. The agreement with numerics is quite good.
We can adopt our approximate solution to study the perturbations of coupled Euler-Poisson-Laplace system describing a self-gravitating system.

We start from exact field equations in Clebsch formalism. (which is a potentials’ formulation for flows not necessarily irrotational).

Clebsch decomposition

\[
\vec{v} = \nabla \chi + \beta \nabla \gamma, \\
\vec{\omega} = \nabla \times \vec{v} = \nabla \beta \times \nabla \gamma.
\]

Euler-Poisson-Laplace field equations

\[
\dot{\rho} + \nabla \cdot (\rho \vec{v}) = 0 \\
\dot{\gamma} + (\vec{v} \cdot \nabla) \gamma = 0 \\
\dot{\beta} + (\vec{v} \cdot \nabla) \beta = 0 \\
\frac{1}{2}v^2 + \dot{\chi} + \beta \dot{\gamma} + h + \Phi = 0 \\
\nabla^2 \Phi_{\text{ext}} = 0. \\
\n\nabla^2 \Phi = 4\pi G \rho
\]

free-boundary problem

\[
h = \frac{du}{d\rho} \text{ represents the specific enthalpy.} \\
u(\rho) \text{ the internal energy density.}
\]
Define a linear perturbation around an exact solution.

Let 

\[ \tilde{v}_1 = \nabla \psi_1 + \tilde{\xi}_1, \]

\[ \psi_1 = \chi_1 + \beta_0 \gamma_1 \]

and

\[ \tilde{\xi}_1 = \beta_1 \nabla \gamma_0 - \gamma_1 \nabla \beta_0 \]

It is useful to introduce the quantity:

\[ c^2 = \left( \frac{dp}{d\rho} \right)_0 = \rho_0 \left( \frac{d^2 u}{d\rho^2} \right)_0, \]

background local speed of sound.
Surprisingly, part of the perturbed Euler-Poisson-Laplace system can be rewritten in this compact four dimensional tensor notation:

\[
\frac{1}{\sqrt{-g}} \partial_{\mu} (\sqrt{-g} g^{\mu \nu} \partial_{\nu} \psi_1) = \frac{c}{\rho_0^2} \left[ \nabla \cdot \left( \rho_0 \ddot{\xi}_1 \right) - \rho_0 \left( \frac{D^{(0)}}{dt} \left( \frac{\Phi_1}{c^2} \right) \right) \right].
\]

\[
\frac{D^{(0)}}{dt} \ddot{\xi}_1 = \nabla \psi_1 \times \dot{\omega}_0 - (\ddot{\xi}_1 \cdot \nabla) \dot{v}_0.
\]

\[
[\nabla^2 + K_j^j] \Phi_1 = -k_j^j \frac{D^{(0)}}{dt} \psi_1.
\]

\[
\nabla^2 \Phi_{\text{ext}} = 0.
\]

**Novel result**

Surprisingly, the linear perturbations of a white dwarf too “know” curved space-time physics!
If we have a space-time metric, we can construct null trajectories (light rays) giving causality. These can be compared with the fully numerically integrated perturbed fluid problem.

**Null rays on the white dwarf acoustic metric.**
These give information on WD perturbation causality

**Numerical integration of the full radial perturbative Newtonian problem with no moving stellar surface (it is kept fixed for simplicity).**
What does it happen if the background solution is allowed?

The study in the case of rotation has been studied by the authors for simplicity in the analytical case of a \( n=1 \) uniformly rotating polytrope analogously (Williams’ solution).

Analytical solutions for a rotating white dwarf are not expected to be simply obtainable. Numerical solutions “à la Eriguchi- Muller” should be implemented then (work in progress).

William’s solution plot for a highly rotating configuration

Acoustic “light rays” for the associated acoustic metric
We have constructed an **approximated analytical solution** for the WD equation with an accuracy for the stellar radius of around 1% in comparison with numerical solutions.

We have applied the **analog gravity formalism** to classical self-gravitating systems described not by a classical polytropic but by a **quantum equation of state**.

In this case too, for totally general (i.e. rotational) perturbations, an **acoustic metric occurs** regulating causality for the classical WD perturbations.

On the **stellar surface**, where density goes to zero, an **infinite acoustic Riemann** tensor develops. It is expected because background matter propagates waves. If matter vanishes, then no waves can exist and effective space-times break down (**useful to understand GR singularities**.)