The solution of Thomas-Fermi equation in presence of a strong magnetic field

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Introduction

• Using the timing observations of radio pulsars, the presence of a magnetic field about $10^{13}\text{G}$ on their surface is established and also in the recent past pulsars with a very high rate of change of period have been observed, implying a high surface magnetic field up to $10^{15}\text{G}$.

• Considering some dimensional analysis for example the scalar virial theorem, the limit of magnetic field at the center of NS is grown up to $10^{18}\text{G}$.

• In order to have the better knowledge of the dynamic and structure NS we must investigate the probable effects of magnetic field.

• After the discovery of the radio pulsar in 1968 and the theoretical proposition of neutron stars, the many paper tried to explain the dynamic and structure of them. Now that is believed, the composed of highly degenerate neutrons, protons, electrons and sometimes muons exit at the core of a NS

• The aim of this talk is to explore the effects of magnetic fields in a broad range of intensities on the properties of a degenerate compressed fluid of neutrons, protons, and electrons in beta equilibrium. We then formulate and solve analytically the Thomas-Fermi equation for these configurations within the Landau theory of energy levels. Such a configuration constitutes the simplest model one can assume for the matter composition of neutron star cores.
Determining of the system

• A global neutral and compressed system of neutrons, protons and electrons in a strong constant average magnetic field is considered.
• The electrons are in ultra-relativistic limit.
• The baryon number, A, is constant.
• The density of system is about normal nuclear density
• The radius of the system is

\[
R_c = r_0 A^{1/3} = \frac{h}{m_v c} \Delta N_p^{1/3}
\]
\[
\Delta = \frac{r_0}{h/m_v c} \left( \frac{A}{N_p} \right)^{1/3}
\]

• The beta equilibrium is valid for the system \( \mu_r = \mu_p + \mu_e \)
• All of protons are confined into a box with Rc radius also the density number of protons is constant.

\[
n_p(r) = \frac{Z}{\frac{4}{3} \pi R_c^3} \theta(r - R_c) = \frac{3}{4 \pi \Delta^3 \chi^3} \theta(r - R_c)
\]
The Thomas-Fermi equation in magnetic fields

Now if we consider \( n_e(0) = n_p(0) \)

\[
C^3 + bC = 1
\]

And boundary conditions are

Next it’s better using \( \dot{\varphi}(\xi) = C - f(\xi) \) where \( f(-x_c) = 0 \)

\[
\dot{\varphi}(\xi) = C - (b + 3C^2)[1 + \left( \frac{b + C^2}{2} \right)^{1/2} \sinh(\beta - \sqrt{b + 3C^2} \xi)]^{-1}
\]

While for \( \infty > \xi > 0 \) and \( E^F_e = 0 \),

\[
\dot{\varphi}(\xi) = \frac{\sqrt{2b}}{\sinh(\sqrt{b} \xi + \Omega)}
\]

Whereas, for non-vanishing Fermi-energy \( E^F_e > 0 \), we have,

\[
\dot{\varphi}(\xi) - \sqrt{-b + \sqrt{b^2 + G}} \left[ \cos \left( am[(b^2 + G)^{1/4}(\xi - 0), 1/2 + b/(2\dot{\varphi}(\xi))] \right) \right]^{-1},
\]

\[
\sinh \beta = \sqrt{\frac{2}{b + C^2} \left[ C - \dot{\varphi}(0) \right] - 1}
\]

\[
\Omega = \frac{\sqrt{2}}{\dot{\varphi}(0)}
\]
The distribution of particles in a strong magnetic field

One can obtain number densities of charge particles in terms of Fermi momentum by using Euler-Maclaurin formula

\[
\begin{align*}
n_q(r) &\approx \frac{1}{3\pi^2\hbar^3} (p_{z,q}^F)^3 \left(1 - \frac{3}{2}(n_{max}^F)^{-1} + (n_{max}^F)^{-3/2}\right) + \frac{eB}{2\pi^2\hbar^2} p_{z,q}^F \left(2 - \frac{1}{2}(n_{max}^F)^{-1} + (n_{max}^F)^{-1/2}\right) \\
&\quad + \frac{(eB)3/2}{2\pi^2\hbar^{3/2}} (-1/12 - 1/720) + \frac{eB^2}{2\pi^2\hbar p_{z,q}^F} (1/12 + (24n_{max}^F)^{-1}) + \tilde{f}(B^{5/2}),
\end{align*}
\]

for \( eB \ll n_q^{2/3}(0) \)

\[
\begin{align*}
n_q(r) &\approx \frac{(p_{z,q}^F)^3}{3\pi^2\hbar^3} \left(1 - \frac{3eB}{2n_q^{2/3}\hbar} + \frac{eB}{n_q^{2/3}\hbar} \right)^{3/2} + \frac{eB p_{z,q}^F}{2\pi^2\hbar^2} \left(2 - \frac{eB}{n_q^{2/3}\hbar} + \frac{eB}{n_q^{2/3}\hbar} \right)^{1/2} \\
n_e(r) &= \eta_1^e \frac{eB}{\pi^2\hbar^2} p_{z,e}^F + \eta_2^e \frac{(p_{z,e}^F)^3}{3\pi^2\hbar^3}, \quad n_p = \eta_1^p \frac{eB}{2\pi^2\hbar^2 c} p_{z,p}^F + \eta_2^p \frac{(p_{z,p}^F)^3}{3\pi^2\hbar^3},
\end{align*}
\]
The Thomas-Fermi equation in magnetic field

The Poisson equation is

\[
\nabla^2 V(r) = -4\pi e [n_p(r) - n_e(r)]
\]

Where

\[
n_p = \frac{N_p}{\frac{1}{3} \pi R_e^3}, \quad n_e(r) = \frac{eB}{2\pi^2 \hbar^2 c} \left[ p_z^F(0) + \frac{2Bc}{3Bm_e^2c^2} \left( p_z^F(0) \right)^3 \right]
\]

In ultra-relativistic limit we have

\[
cp_{z, electron}^F(0) = E_e^F + eV(r)
\]

Next

\[
\nabla^2 V(r) = -4\pi e [n_p(r) - \frac{eB}{2\pi^2 \hbar^2 c^2} eV(r) - \frac{1}{3\pi^2 \hbar^3} (eV(r)/c)^3]
\]
The Thomas-Fermi equation in magnetic fields

To solve the Thomas–Fermi equation in magnetic field, that is better we use below dimensionless quantities

\[ \kappa = \frac{1}{\sqrt{4\pi \alpha}} \frac{\hbar}{m_e c} \left[ \frac{4\pi}{3} \Delta^3 \right]^{-1/3} \left[ \frac{1}{3\pi^2} \right]^{1/6} \]
\[ \varphi = \left[ \frac{3}{4\pi \Delta^3} \right]^{-1/3} \left[ \frac{1}{3\pi^2} \right]^{1/3} \frac{\hbar}{m_e} \frac{eV + E_F}{hc} \]

\[ r = \kappa x, \quad R_C = \kappa x_c \]

And also we define \( \dot{\varphi}(\xi) = \varphi(\xi + x_c) \), and \( b = \frac{B}{B_\pi} \left[ \frac{2}{3\pi^2} \right]^{1/3} \Delta^2 \). After some calculations

\[ \varphi''(\xi) = -\theta(-\xi) + b\dot{\varphi}(\xi) + \dot{\varphi}(\xi)^3 \]

One can multiply \( \dot{\varphi} \) in the above equation and integrate on the both sides of it

\[ 2[\dot{\varphi}'(\xi)]^2 = \begin{cases} -4\dot{\varphi} + 2b\dot{\phi}^2 + \dot{\varphi}^4 + a, & \xi < 0; \\ +\dot{\varphi}^4 + 2b\dot{\phi}^2 - \dot{\varphi}^4(\xi^{WS}) - 2b\dot{\phi}^2(\xi^{WS}), & \xi > 0. \end{cases} \]

Here we use \( \dot{\varphi}'(\xi^{WS}) = 0 \) and also the electric field vanish in the center of system

\[ a = 4C - C^4 - 2bC^2; \quad C = \dot{\varphi}(-x_c) \]
The Thomas-Fermi equation in magnetic fields

Now if we consider \( n_e(0) = n_p(0) \)

\[
C^3 + bC - 1
\]

And boundary conditions are

\[
\dot{\varphi}(0) = \frac{\left( a + \dot{\varphi}^4(\xi_0) + 2b\dot{\varphi}^2(\xi_0) \right) / 4}{\sqrt{\frac{\dot{\varphi}^4(0)}{2} + 2b\dot{\varphi}^2(0) - \frac{\dot{\varphi}^4(\xi_0)}{2} - 2b\dot{\varphi}^2(\xi_0)}}
\]

Next, it's better using \( \dot{\varphi}(\xi) = C - f(\xi) \) where

\[
f(-x_c) = 0
\]

For \( -x_c < \xi < 0 \)

\[
\dot{\varphi}(\xi) = C - (b + 3C^2)[1 + \left( \frac{b + C^2}{2} \right)^{1/2} \sinh(\beta - \sqrt{b + 3C^2}\xi)]^{-1}
\]

While for \( \infty > \xi > 0 \) and \( E_e^F = 0 \),

\[
\dot{\varphi}(\xi) = \frac{\sqrt{2b}}{\sinh(\sqrt{b}\xi + \Omega)}
\]

Whereas, for non-vanishing Fermi-energy \( E_e^F > 0 \), we have,

\[
\dot{\varphi}(\xi) = \sqrt{b + \sqrt{b^2 + C^2}} \left[ \cos \left( a m \{(b^2 + C^2)^{1/4}(\xi - \Theta), 1/2 + b/(2\dot{\varphi}^2(\xi_0)) \} \right) \right]^{-1},
\]

\[
\sinh \beta = \sqrt{\frac{2}{b + C^2}} \left[ \frac{b + 3C^2}{C - \dot{\varphi}(0)} - 1 \right]
\]

\[
\Omega = \frac{\sqrt{2}}{\dot{\varphi}(0)}
\]
The Thomas-Fermi equation in magnetic fields

For $B \to 0 \Rightarrow \begin{cases} b \to 0 \\ C \to 1 \\ \phi \bigg|_{B \neq 0} \to \phi \bigg|_{B=0} \end{cases}

\hat{\phi}(\xi) = \begin{cases} 1 - 3 \left[ 1 + 2^{-1/2} \sinh(a - \sqrt{3}\xi) \right]^{-1} , & \xi < 0 \\ \frac{\sqrt{2}}{\xi + b} , & \xi > 0 \end{cases}

a = \arcsinh(11\sqrt{2}) = 3.439, \quad b = \left(\frac{4}{3}\right)\sqrt{2} = 1.886.

The Thomas-Fermi equation in magnetic fields

Coulomb potential and electric field

\[ eV(\xi) = \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{\Delta} m_\pi c^2 \tilde{\phi}(\xi) - E^F \]

\[ E(\xi) = \left( \frac{3^5\pi}{4} \right)^{1/6} \frac{\sqrt{\alpha}}{\Delta^2} \frac{m_\pi^2 c^3}{\epsilon h} \tilde{\phi}'(\xi) \]

the coulomb potential at the center of the configuration and the electric field at the surface of the core are:

\[ eV(0) = \left( \frac{9\pi}{4} \right)^{1/3} \frac{1}{\Delta} m_\pi c^2 C - E^F \]

\[ E_{\text{max}} = \epsilon_0 \frac{\sqrt{\alpha}}{\Delta^2} \left( \frac{m_\pi}{m_e} \right)^2 E_c \]

\[ \epsilon_0 = \left( \frac{3^5\pi}{4} \right)^{1/6} \tilde{\phi}'(0) \]

These physical quantities depend on the parameter \( \Delta \) [or the proton number-density \( A/N_p \) and mass density], magnetic field \( B \).

\[ \Delta \approx \left( r_0/\lambda_\pi \right) \left( A/N_p \right)^{1/3} \]
The estimation of the ratio $A/N_p$

\[ n_e(r) = \eta_1^e \frac{eB}{\pi^2 \hbar^3} p_{z,e}^F + \eta_2^e \frac{(p_{z,e}^F)^3}{3\pi^2 \hbar^3}, \]

\[ n_p = \eta_1^p \frac{eB}{2\pi^2 \hbar^2 c} p_{z,p}^F + \eta_2^p \frac{(p_{z,p}^F)^3}{3\pi^2 \hbar^3} \]

From Beta equilibrium

\[ \mu_n = \mu_p + \mu_e \]

\[ \sqrt{(3\pi^2 \hbar^3 c^3 \bar{n}_n)^{2/3} + m_n^2 c^4} = \sqrt{(3\pi^2 \hbar^3 c^3 n_p)^{2/3} C^2 + m_p^2 c^4} + (3\pi^2 \hbar^3 c^3 n_p)^{1/3} C. \]

Also by assuming $A=\text{constant} \approx 10^{57}$

\[ A = 4\pi \int_0^{R_c} r^2 (n_p + \bar{n}_n) dr \]

\[ b = \frac{B}{B_c} \left[ \frac{2}{3\pi^2} \right]^{1/3} \Delta^2. \]
The estimation of the ratio $A/N_p$

Magnetic field cases total number of charge particles increase and total number of neutrons decrease

\[
\frac{A}{N_p} \bigg|_{b \neq 0} = \frac{A}{N_p} \bigg|_{b = 0} C^3, \quad \Delta \bigg|_{b \neq 0} = \Delta \bigg|_{b = 0} C
\]

\[
C^3 + bC = 1
\]

\[
b = \tilde{b}C^2 = \left[\frac{2}{3\pi^2}\right]^{1/3} (\Delta \bigg|_{b = 0})^2 \frac{B}{B_\pi} C^2
\]

\[
C^3 = \frac{1}{1 + \tilde{b}}
\]

For magnetic field $B \approx 10^{15}$ G, $A/N_p$ decrease about 4-5%
Fermi momenta of charge particles in magnetic fields

The Fermi momentum of electrons and protons at the core are not very sensitive by magnetic field.
Coulomb potential and electric field in magnetic field

\[ E_{e}^{F} \approx 0.1m_{\pi}c^{2} \]
Coulomb potential and electric field in magnetic field

\[ E^F_e \approx 0.1 \mu \pi e^2 \]
The number densities of protons and electrons

\[ n_e \bigg|_{B \neq 0} \approx n_e \bigg|_{B = 0} (1 + \tilde{b}) \]

\[ n_p \bigg|_{B \neq 0} \approx n_p \bigg|_{B = 0} (1 + \tilde{b}) \]
The Fermi energy of electron & magnetic field

If we consider \( eV(\xi^{WS}) \sim 0 \)

\[
e V(\xi) = (3\pi^2 h^3 c^3 n_p)^{1/3} (\phi(\xi) - \phi(\xi^{WS})) \quad \phi(\xi^{WS}) \approx \Delta \left( \frac{4}{9\pi} \right)^{1/3} \frac{E^F}{m_\pi c^2}
\]

If we assume \( \hat{\phi}(\xi^{WS}) = C \Rightarrow \hat{\phi}(0) = \hat{\phi}(-x_c) = C \)

It means the distribution of electron is constant and we have local neutrality for over all system. We can define

\[
E_{e,max}^F = (3\pi^2 h^3 n_p)^{1/3} C
\]

By this energy Fermi of electrons, all electrons are confined in the box with Rc radius.

\[
E_{e,max}^F \equiv \left( \frac{9\pi}{4} \right)^{1/3} \frac{C}{\Delta_{b\neq 0}} m_\pi c^2,
\]

\[
= \left( \frac{9\pi}{4} \right)^{1/3} (\Delta_{b=0})^{-1} m_\pi c^2,
\]
we exactly obtain analytical solutions for different values of magnetic fields $B$, much larger than $10^{13}$ Gauss, but smaller than $10^{17}$ Gauss.

- The number density of protons and electrons increase in magnetic field.

\[
\begin{align*}
\left. n_e \right|_{B \neq 0} & \approx \left. n_e \right|_{B = 0} (1 + \tilde{b}) \\
\left. n_p \right|_{B \neq 0} & \approx \left. n_p \right|_{B = 0} (1 + \tilde{b})
\end{align*}
\]

- The ratio $\lambda_{e,p}$ increases with magnetic fields.

\[
\lambda_{e,p}(B) = (1 + \tilde{b}) \times \lambda_{e,p}(B = 0)
\]

- Maximum Fermi energy of electron which is necessary to confine all of electrons into nucleon core $r < R_c$ doesn’t change in magnetic field.

- The value of electric field at the near surface increase by magnetic field.