Where is $\hbar$ Hiding in Entropic Gravity?

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P. Chen and C-H. Wang, arXiv:1112.3078 [gr-qc].
Preamble

• The relationship between gravity and thermodynamics has been investigated for decades since Bekenstein’s seminal discovery of the BH area law and Hawking’s BH evaporation. A result of ‘entanglement entropy’

• Inversion of logic: gravity an emergent phenomenon induced by QF fluctuations (Sakharov 68); Einstein eq. as an equation of state from TD (Jacobson 95), etc.

• E. Verlinde 11: gravity as an entropic force (like elastic polymer). Max. entropy Newton’s law.
• Verlinde’s Key ingredients:
  1. $F dx = T dS.$ and $dS$ associated with $\lambda_c$ (Compton w.)
  2. Holographic principle & equipartition theorem for T.

• Both involve $\hbar$, i.e., QM in nature. Yet all $\hbar$‘s perfectly cancelled pure classical Newton’s law emerged.

• Atypical for emergent phenomena in physics. So, where is $\hbar$ hiding in entropic gravity?

• We argue that when dealing with quantum gravity, GUP (generalized uncertainty principle (Veneziano 86, Gross-Mende 87)) is the more appropriate foundation.

• Based on GUP, BH Bekenstein entropy area law must be modified in both strong & weak gravity (Adler, Chen, Santiago 01).
Holographic Principal and Holographic Entanglement Entropy

- BH entropy proportional to its area:

\[ S_B = \frac{4\pi k_B GM^2}{hc} = 4\pi \frac{M^2}{M_P^2} = \frac{k_B c^3}{4hG} A. \]

- Holographic principle:

Information content \( S \) of an enclosed spacetime region should be no larger than the Bekenstein-Hawking entropy.

\[ S \leq \frac{k_B}{4L_P^2} A = \frac{k_B c^3}{4hG} A = S_B. \]

- Note: Equal-sign was used in EG; unjustified.
Holographic Principal and Holographic Entanglement Entropy

• Holographic Entanglement Entropy

Correlation between subsystem A and its complementary subsystem B.

Hilbert space: \( H_{tot} = H_A \otimes H_B \).

If an observer can access entire space, then

\[ H_S = -k_B \sum_i P_i \ln(P_i), \]

i.e., the von Neumann entropy in \( H_{tot} \) with density matrix \( \rho_{tot} \) is

\[ S(\rho_{tot}) = -k_B \text{Tr}(\rho_{tot} \ln \rho_{tot}). \]

• If accessible only to A, then \( \rho_A = \text{Tr}_B \rho_{tot} \).
• Entanglement entropy (EE) is thus defined as the von Neumann entropy for reduced matrix $\rho_A$:

$$S_A = -k_B Tr_A (\rho_A \ln \rho_A).$$

• If the total state is entangled, i.e., it cannot be factorized as $|\Psi_{tot}\rangle = |\Psi_A\rangle \otimes |\Psi_B\rangle$, then EE is non-vanishing.

• For pure states, $S_A = S_B$. (Srednicki)

$$S_E = \frac{\text{Area}(\Sigma)c^3 k_B}{4\hbar G} + \text{subleading terms}.$$ 

• True for min. surface, i.e., 0 extrinsic curvature. Also true for 2-sphere and 2-cylinder.
Verlinde’s Entropic Gravity Scenario

• Verlinde's system: a massive source $M$ is encoded by a spherical screen with radius $R$, and test particle $m$ is placed just outside the screen.

• Holographic screen: Test particle interact with screen. If entropy varies as test particle moves, then it will front a restoring force according to $Fdx = TdS$.

• Verlinde invokes HP with maximum entropy, which is only true for BH horizon, for the screen, which is problematic.
Entropy Variation Law

• Verlinde’s Approach

“When a particle is one Compton wavelength from the horizon, it is considered to be part of BH” (Bekenstein).

• Verlinde:

\[ \Delta S = -2\pi k_B \frac{\Delta x}{\lambda_m} = -2\pi k_B \frac{mc}{h} \Delta x. \]

• QM dictates such uncertainty. So how would the horizon react to this infinitesimal displacement?

• Further challenge: \( \Delta S \) and \( S = A / 4L_P^2 \) may not be compatible a priori, but they should:

\[
\Delta S = \frac{k_B c^3}{4hG} \Delta A = -2\pi k_B \frac{mc}{h} \Delta x \quad \Rightarrow \quad \Delta A = -8\pi \frac{mG}{c^2} \Delta x
\]
Fursaev's system: two infinite surface $B_1$ and $B_2$, with their z coordinates fixed, are placed around a massive source $M$. Outside the sphere is a test particle $m$ whose displacement will affect the area of the surfaces.
Entropy Variation Law

• Fursaev’s Approach

Spacetime metric in weak gravity:

\[ ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 + \frac{2GM}{rc^2}\right)dr^2 + r^2 d\Omega^2 \]

\[ = -\left(1 - \frac{2GM}{\rho c^2}\right)c^2 dt^2 + \left(1 + \frac{2GM}{\rho c^2}\right)(dx^2 + dy^2 + dz^2), \]

where \( \rho = r(1 - GM / rc^2) = \sqrt{x^2 + y^2 + z^2} \).
Entropy Variation Law

• Fursaev’s Approach

Presence of test mass $m$ causes a back-reaction:

$$ds^2 = -\left(1 - \frac{2GM}{c^2 \rho} - \frac{2Gm}{c^2 \rho_0}\right)c^2 dt^2$$

$$+ \left(1 + \frac{2GM}{c^2 \rho} + \frac{2Gm}{c^2 \rho_0}\right)(dx^2 + dy^2 + dz^2)$$
Entropy Variation Law

• Fursaev’s Approach

• A small segment of area on one infinite surface is

\[ da^2 = g_{xx} dx^2 g_{yy} dy^2. \]

The total surface area is therefore

\[ A_k = \int \int dx \, dy \left( 1 + \frac{2GM}{c^2 \rho_k} + \frac{2Gm}{c^2 \rho_{k,0}} \right). \]

as distance between test particle and the surface changes by an amount \( \Delta r \), area will change by

\[ \Delta A_k = -4 \pi Gm \Delta r / c^2. \]

The total variation is therefore

\[ \Delta A = \Delta A_1 + \Delta A_2 = -\frac{8 \pi Gm \Delta r}{c^2}. \]
Our system: a massive source $M$ is encoded by a spherical screen with radius $R$, and test particle $m$ is placed at a distance $r_0$ outside the screen.
Entrophy Variation Law

- **Our Approach**

Although Fursaev successfully reproduces entropy variation law consistent with Bekenstein entropy, this derivation is only valid for infinite surface. The more physically relevant geometry should be a sphere, on which temperature can be well-defined.

\[
ds^2 = -\left(1 - \frac{2GM}{rc^2}\right)c^2 dt^2 + \left(1 + \frac{2GM}{rc^2}\right)(d\rho^2 + \rho^2 d\Omega^2).
\]

\[\rho_0 < \rho.\]

\[A = 4\pi\rho^2 - \frac{8\pi GM \rho}{c^2} + A_{m},\]
Entropy Variation Law

• Our Approach

Change of surface area due to the presence of test particle is

\[ A_m = \begin{cases} 
\frac{8\pi Gm\rho^2}{c^2\rho_0}, & \rho_0 > \rho, \\
8\pi Gm\rho \quad \text{c^2}, & \rho_0 < \rho.
\end{cases} \]

Keeping the leading orders of \( GMR/c^2 \) and \( GmR/c^2 \):

\[ \begin{cases} 
r_0 > R : A = 4\pi R^2 + \frac{8\pi GmR^2}{c^2 r_0}, \\
r_0 < R : A = 4\pi R^2 - \frac{8\pi GmR}{c^2}.
\end{cases} \]
Entropy Variation Law

• Our Approach

Now when the test particle moves, the surface area will change by an amount

\[
\frac{\partial A}{\partial r_0} \Delta r_0 = -\frac{8 \pi Gmr^2}{c^2 r_0^2} \Delta r_0.
\]

• Following Bekenstein law, the entropy variation is

\[
\Delta S = k_B \frac{\Delta A}{4 L_P^2} = -\frac{2 \pi k_B r^2}{r_0^2} \frac{mc}{h} \Delta r_0.
\]

• When the test particle is just outside the sphere,

\[
\Delta S = -2 \pi k_B \frac{mc}{h} \Delta r_0 = -2 \pi k_B \frac{\Delta r_0}{\lambda_c}.
\]
Temperature

• As a consequence of entanglement entropy that obeys Bekenstein law, degree of freedom on screen is

\[ N = Ac^3 / Gh. \]

Under equipartition rule,

\[ E = Mc^2 = \frac{1}{2} Nk_B T. \]

Therefore

\[ T = \frac{2Mc^2}{Nk_B} = \frac{2Gh}{ck_B} \frac{M}{A}. \]

Following the first law of TD, \( F \Delta x = -T \Delta S. \) we find

\[ F = -GMm / R^2. \]
To uncover the missing QM contribution in EG, we now invoke GUP. Based on GUP, BH temperature has the form

\[ T_{GUP} = \frac{M c^2}{4 \pi k_B} \left[ 1 - \sqrt{1 - \frac{M_P^2}{M^2}} \right]. \]

In the limit \( M_P / M = 1 \),

\[ T_{GUP} = \frac{M_P^2 c^2}{8 \pi k_B M} \left[ 1 + \frac{M_P^2}{4 M^2} + \frac{M_P^4}{8 M^4} + \ldots \right], \]

The corresponding BH entropy is thus

\[ S_{GUP} = 2 \pi k_B \left\{ \frac{M^2}{M_P^2} \left[ 1 - \frac{M_P^2}{M^2} + \sqrt{1 - \frac{M_P^2}{M^2}} \right] - \log \left[ \frac{M}{M_P} \left( 1 + \sqrt{1 - \frac{M_P^2}{M^8}} \right) \right] \right\}. \]
Generalized Uncertainty Principle

- In the large BH mass limit,

\[
S_{GUP} = 4\pi k_B \frac{M^2}{M_P^2} - \pi k_B \log \left( \frac{M^2}{M_P^2} \right) + \text{const}...
\]

\[
= k_B \frac{A}{4L_P^2} - \pi k_B \log \left( \frac{A}{L_P^2} \right) + \text{const}...
\]

- The BH entanglement entropy is then

\[
S_{GUP} = \frac{Ak_B}{8L_P^2} \left[ 1 - \frac{16\pi L_P^2}{A} + \sqrt{1 - \frac{16\pi L_P^2}{A}} \right]
\]

\[
-2\pi k_B \log \left[ \frac{A}{4\sqrt{\pi L_P}} \left( 1 + \sqrt{1 - \frac{16\pi L_P^2}{A}} \right) \right].
\]
Quantum Effects in Entropic Gravity

• The entropy variation law is directly affected by GUP:

\[ \Delta S = \frac{\partial S_{GUP}}{\partial A} \Delta A, \]

• Under GUP,

\[ N = \frac{S_{GUP}}{4k_B}. \]

• Once again applying the equipartition formula to determine the temperature on the screen, we find

\[ T = \frac{2Mc^2}{Nk_B} = \frac{Mc^2}{2S_{GUP}}. \]
Quantum Effects in Entropic Gravity

• Finally, invoking the 1\textsuperscript{st} law of TD, we arrive at a modified gravity force:

\[
F_{GUP} = F_N \frac{2\left[\alpha (1 + \eta) - 2(2 + \eta)\right]}{\eta(1 + \eta)\left\{-4 + \alpha (1 + \eta) - 4 \log\left[\alpha (1 + \eta)/2\right]\right\}}.
\]

• In the large distance limit,

\[
F_{GUP} = F_N \left\{ + \alpha \left[2 - \log \alpha\right] + \alpha^2 \left[ 4 - 5 \log \alpha + (\log \alpha)^2 \right] + \ldots \right\}.
\]

\[
\eta = \sqrt{1 - 4L_p^2 / R^2} \equiv \sqrt{1 - 4\alpha}.
\]
Summary

• We argued that the perfect cancellation of would be broken if more exact form of entanglement entropy based on GUP is invoked.
• Based on this we found, in the weak gravity limit, that the hided $\hbar$ in the form of log corrections to Newton’s law.
• However from the purpose of emergent gravity this approach is still unsatisfactory: How do we know the spacetime is warped with RG metric?
• This QM correction to Newton gravity may serve as a probe to entropic gravity.