Late-time expansion in the semiclassical theory of the Hawking radiation

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The present contribution is concerned with the computation of the backreaction effects on the Hawking radiation, in the semiclassical approximation.

1) Hawking radiation (Hawking: Comm. Math. Phys. 43 (1975); E 46 (1976))

\[
F(\omega) d\omega = \frac{1}{2\pi} \Gamma(\omega) \frac{1}{e^{\frac{8\pi GM\omega}{l_P^2}} - 1} d\omega
\]  

(1)

\(\Gamma(\omega)\) is the gray-body factor and \(l_P^2 = \frac{G\hbar}{c^3}\), \(l_P = 1.62 \times 10^{-35} m\). Temperature

\[
T = \frac{l_P^2 c^4}{8\pi GM} = \frac{\hbar c^3}{8\pi G M}
\]

2) External field approach

Quantize a (massless) scalar field in the background gravitational field generated by the black-hole (known procedure due to the existence of a time like Killing vector). Definition of the physical vacuum which turns
out not to be the “Killing” or Boulware vacuum and thus a stationary observer at space infinity detects a radiation.

\[
\phi = \int_0^\infty \frac{d\omega}{\sqrt{2\omega}} (u_\omega(\hat{r})e^{-i\omega t}a(\omega) + u_\omega^*(\hat{r})e^{i\omega t}a^+(\omega)) = \\
= \int_0^\infty \frac{dk}{\sqrt{2k}} (v_k(\hat{r}, t)b(k) + v_k^*(\hat{r}, t)b^+(k)).
\]

Unruh vacuum is

\[b(k)|0_u\rangle = 0\]

where \(b(k)\) are the annihilation operators for certain modes which are regular at the horizon. Under such condition the falling detector does not experience excitations. Knowledge of \(u\) and \(v\) allows to compute the Bogoliubov transformation

\[a(\omega) = \int_0^\infty dk (\alpha_{\omega k} b(k) + \beta_{\omega k} b^+(k))\]

from which one can compute the average value of the number operator
on the Unruh (true) vacuum

\[ N(\omega) = \langle 0_u | a^+(\omega) a(\omega) | 0_u \rangle = \int_0^\infty dk |\beta_{\omega k}|^2 \]

Work outside the horizon. Again scalar field (simpler if massless). Construct the local observable

\[ Q^+ Q \]

where

\[ Q = \int \sqrt{-g} \phi(x) h(x) d^4x \]

\( h \) test function with 4-dimensional compact support. \( Q_T \) is \( Q \) with the support translated by a large time \( T \). They compute such a mean value by piling up (propagating back) the field on a space like surface \( \tau = const \). The result is that

\[ \langle 0| Q^+_T Q_T |0 \rangle \]

is expressed (rigorously for free fields) in terms of the Wightman (not Feynman) function

\[ W(x, x') = \langle 0| \phi(x) \phi(x')|0 \rangle \]
\[ \tau = 0 \]

\[ r = \text{const.} \]

Kruskal–Szekeres coordinates
computed on $\tau = const$, thus at very short distances and very near the future event horizon.

$$W(x, x') = \frac{1}{4\pi} \frac{1}{\sigma \varepsilon} + \text{lower terms}$$

in Minkowski would be

$$W(x, x') = \frac{1}{4\pi} \frac{1}{(x - x')^2 - (x^0 - x'^0 - i\varepsilon)^2}$$

Takes the place of the definition of the vacuum.

Independent of $T$ (time): The Hawking radiation is a permanent phenomenon.
3) Semiclassical treatment

Introduced by Kraus and Wilczek (Nucl.Phys. B433 (1995)).
Work out a dynamical model which takes into account conservation of energy.
The model (4-dimensions): a spherical shell of matter (massive or massless) moving in the gravitational field generated by the black-hole and by the shell itself (only s-waves).
Compute the modes (regular and singular) semiclassically i.e.

\[ e^{iS/\hbar} \]

where \( S \) is the classical action of the shell moving in the gravitational field. Use such modes to compute the Bogoliubov coefficients.
The classical problem is exactly soluble even if non trivial.

Choice of gauge and the conjugate momentum

In the general expression of the metric (ADM)

$$ ds^2 = -N^2 dt^2 + L^2 (dr + N^r dt)^2 + R^2 d\Omega^2 $$

(2)

all quantities $N, L, N^r, R$ are supposed functions only of the radial variable $r$ and $t$ thus realizing spherical symmetry.

Painlevé-Gullstrand metric characterized by setting $L = 1$. Such a metric has the advantage of being non singular at the horizon. After fixing $L = 1$ one has still a gauge choice on $R$. In presence of a shell of matter one cannot choose $R = r$. One has several choices (F.Fiamberti, P.M. Nucl.Phys. B 794 (2008); P.M. Class.Q.Gravity 27 (2010))

We will use the “outer gauge” which is defined by $R = r$ for $r \geq \hat{r}$ where $\hat{r}$ denotes the shell position. At $r = \hat{r}$, $R$ is continuous as all the other functions appearing in (2), but its derivative is discontinuous.
outer gauge

R

\[ \hat{r} \quad r \]
After solving the constraints one reaches the reduced action i.e. a form in which only the coordinate $\hat{r}$ of the shell and a conjugate momentum appears, in addition to the boundary terms. The reduced action in the outer gauge is given by (F. Fiamberti P.M. Nucl.Phys. B 794 (2008); P.M. Class.Q.Gravity 27 (2010))

$$S = \int_{t_i}^{t_f} \left( p_c \dot{\hat{r}} - \dot{M}(t) \int_{r_i}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr - H N(r_e) + M N(r_i) \right) dt \quad (3)$$

With $M$ as a datum and the normalization $N(r_e) = 1$ is equivalent to

$$S = \int_{t_i}^{t_f} (p_c \dot{\hat{r}} - H) dt. \quad (4)$$

In the massless case (massive far more complicated)

$$p_c = \sqrt{2M\hat{r}} - \sqrt{2H\hat{r}} - \hat{r} \log \frac{\sqrt{\hat{r}} - \sqrt{2H}}{\sqrt{\hat{r}} - \sqrt{2M}}. \quad (5)$$

$H$ as implicit function (invertible) of $\hat{r}$ and $t$. One has to keep in mind that $p_c$ is not the kinetic momentum of the shell but the conjugate momentum with respect to $\hat{r}$ of the whole system.
The action for the modes regular at the horizon

In the following we use the symbol $r$ for $\hat{r}$.
At the semiclassical level the modes which are invariant under the Killing vector $\frac{\partial}{\partial t}$ are simply given by

$$e^{iS/l_P^2}$$

with $l_P^2 = G\hbar$ the square of the Planck length and

$$S = \int_{r_1}^{r_1} p_c dr - Ht + \text{const.}$$

Such modes have the feature of being singular at the horizon; this is immediately seen from the expression of $p_c$ eq.(5) which diverges at $r = 2H$. The vacuum given by $a_\omega |0\rangle = 0$, being $a_\omega$ the destruction operator relative to the described modes gives rise to a singular description at the horizon, while a free falling observer should not experiment any singularity. They describe your detector.
Construction of regular (Unruh) modes

i) at time 0 the conjugate momentum is a given value $k$;
ii) at time $t$ the shell position $r$ is a given value $r_1$.

Structure already given by Kraus and Wilczek.

With the two conditions $p_c(0) = k$ and $r(t) = r_1$ the action is

$$S(r_1, t, k) = kr_0(r_1, t, k) + \int_0^t (p_c \dot{r} - H(r(t'), p_c(t'))) dt' =$$

$$= kr_0(r_1, t, k) + \int_0^t p_c \dot{r} dt' - H[r_1, t, k] t. \quad (8)$$

The last equality is due to the fact that $H$ along the motion is a constant despite $H$ depends on the boundary conditions as explicitly written.

$$S(r, 0, k) = kr$$

$r_0$ denotes the value of $r$ at time 0; also such a quantity depends on the imposed boundary conditions. Taking into account that $r$ and $p_c$
depend both on the final time \( t \) and on the running time \( t' \), and denoting with a dot the derivative with respect to \( t' \) one has

\[
\frac{\partial S}{\partial r_1} = k \frac{\partial r_0}{\partial r_1} + \int_0^t (p_c \frac{\partial \dot{r}}{\partial r_1} + \dot{p}_c \frac{\partial r}{\partial r_1}) dt' = p_c. \tag{9}
\]

Similarly

\[
\frac{\partial S}{\partial t} = k \frac{\partial r_0}{\partial t} + (p_c \dot{r} - H)\big|_t + \int_0^t (p_c \frac{\partial \dot{r}}{\partial t} + \dot{p}_c \frac{\partial r}{\partial t}) dt' = -H. \tag{10}
\]

The action (8) has to be computed on the solution of the equation of motion, satisfying the described boundary conditions.

The equations of motion

In the outer gauge, that we adopt here, the equation of motion for \( r \) has the form

\[
\frac{dr}{dt} = 1 - \sqrt{\frac{2H}{r}} \tag{11}
\]
while $dp_c/dt$ can be obtained substituting $\dot{r}(t)$ in eq.(5). Eq.(11) can be integrated in the form

$$t = 4H \log \frac{\sqrt{r_1} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2H}} + r_1 - r_0 + 2\sqrt{2Hr_1} - 2\sqrt{2Hr_0}.$$  \hspace{1cm} (12)$$

The boundary condition at $t = 0$ gives

$$0 < k = \sqrt{2Mr_0} - \sqrt{2Hr_0} - r_0 \log \frac{\sqrt{r_0} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2M}}$$  \hspace{1cm} (13)$$

where $2M < 2H < r_0 < r_1$; eq.(19) together with eq.(18) should determine completely the motion.

*Given* $t$, $k$ and $r_1$: determine $r_0(k, t)$ and $H(k, t)$.

**The occurrence of caustics (Lagrangian singularities)**

(P.M. arXiv:1107.3312 [hep-th])
It is very easy to show that the standard variational problem in which $r$ is fixed to $r_0$ at time 0 and to $r_1$ at time $t$ presents no caustics. To investigate the occurrence of caustics we shall compute the derivative of $t$ with respect to $r_0$ under the constraint of constant $k$.

$$\left(\frac{\partial t}{\partial r_0}\right)_k = \left(\frac{\partial t}{\partial r_0}\right)_H + \left(\frac{\partial t}{\partial H}\right)_{r_0} \left(\frac{\partial H}{\partial r_0}\right)_k = -\frac{1}{1 - \sqrt{a}} [1 - I_1 I_2]$$  \hspace{1cm} (14)

with $a = \frac{2H}{r_0}$.

$$I_1 = (1 - \sqrt{a}) \int^{\sqrt{a}}_{\sqrt{\frac{2M}{r_0}} \frac{y^2}{(1 - y)^2}}$$, \hspace{1cm} $I_2 = (1 - \sqrt{a}) \int^{\sqrt{a}}_{\sqrt{\frac{2H}{r_1} \frac{z^2}{(1 - z)^2}}}$

(15)

One proves that for any given $k$ (Unruh frequency), for large enough $r_1$ the derivative (14), when $r_0$ moves from $r_1$ to $2M$ changes sign, thus vanishing at at least one intermediate point. This implies the occurrence of Lagrangean singularities (caustics). (Arnold: Mathematical methods of classical mechanics)
However analytically one can prove: for $r_1 < 10M$ no caustics; (numerically for $r_1 < 24M$).

Also: Given a $k$ for sufficiently large $t$ no caustics arises, for any $r_1$.

Kraus and Wilczek proposed to perform the time Fourier analysis at a point $r_1$ not too far from the horizon, the reason being that there one should expect the semiclassical approximation to be reliable. We showed above that for $r_1 < r_c$ there are no ambiguities in the definition of the action and in addition it is well known the time Fourier transform gives results independent of $r_1$; thus we shall work with $r_1 < r_c$.

**The late-time expansion**

(Kraus-Wilczek)
Expand the modes at late times
Compute the Bogoliubov coefficient by means of time Fourier transform.
\[
F(\omega)d\omega = \frac{d\omega}{2\pi} \frac{1}{e^{8\pi \frac{M\omega}{l_P^2}(1-\frac{\omega}{M})} - 1} = \frac{d\omega}{2\pi} \frac{1}{e^{8\pi \frac{\omega}{l_P^2}(M-\omega)} - 1} \quad (16)
\]

Repeated by Kraus and Keski-Vakkuri (Nucl.Phys. B491 (997)) with a different method to find
\[
F(\omega)d\omega = \frac{d\omega}{2\pi} \frac{1}{e^{8\pi \frac{M\omega}{l_P^2}(1-\frac{\omega}{2M})} - 1} . \quad (17)
\]

Take up again the late time expansion

(P.M. Phys.Rev. D85, 084005 (2012); arXiv:1107.3312 [hep-th])

\[
t = 4H \log \frac{\sqrt{r_1} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2H}} + r_1 - r_0 + 2\sqrt{2Hr_1} - 2\sqrt{2Hr_0}. \quad (18)
\]

The boundary condition at \( t = 0 \) gives

\[
0 < k = \sqrt{2Mr_0} - \sqrt{2Hr_0} - r_0 \log \frac{\sqrt{r_0} - \sqrt{2H}}{\sqrt{r_0} - \sqrt{2M}} \quad (19)
\]
Introduce now an implicit time variable 
\[ T = \exp\left(-\frac{t}{4H}\right) \]
which due to the bounds on \( H \), for \( t \to +\infty \) tends to 0. It is possible to show that \( H(t) \) is a decreasing function of \( t \); thus the above relation can be inverted \( t = t(T) \) and we can write
\[ h \equiv \sqrt{2H} = \sqrt{2M} + g(T) \equiv m + g(T) \]  \((20)\)

One can prove the following properties of \( g(T) \)

\[ g'(T) > 0, \quad g(0) = 0, \quad g(1) = \sqrt{2H_1} - m, \quad g(T) = c_H^0 T + c_H^1 T^2 + \cdots \] \((21)\)

where \( H_1 \) is the solution of \( p_c(r_0 = r_1) = k \). One can solve eq.(20) by iteration starting with \( h_0 = \sqrt{2M} \) and by induction one proves that
\[ h_{n+1} > h_n \] \((22)\)

and as \( h \) is bounded by \( \sqrt{2H_1} \) the series converges for all \( 0 < t < \infty \).
Figure 2: Time development of $2H(t)$ and $r_0(t)$
We give in Fig.2 a qualitative graph of the behavior in time of $2H(t)$ and $r_0(t)$.

We have for the first two terms of such an iteration procedure

$$H(t) = M + \sqrt{2M} c_0^H \tau + \frac{t}{2M} (c_0^H \tau)^2, \quad \tau = e^{-\frac{t}{4M}} \quad (23)$$

sufficient to give the $O(\omega/M)$ corrections to the Hawking distribution.

Due to eq.(10) the time dependence of the mode which is regular at the horizon, for fixed $r_1$ is (recalling $\partial S/\partial t = -H$ )

$$S = f(r_1) - \int H(t')dt' = \text{const} - Mt + 4M \sqrt{2M} \tau_1 + t\tau_1^2 \quad (24)$$

i.e. for the semiclassical mode we have

$$e^{iS/l_P^2} = e^{i[q(r_1) - Mt + 4M \sqrt{2M} \tau_1 + t\tau_1^2]/l_P^2} \quad (25)$$

where $l_P^2 = G\hbar$ is the square of the Planck length and

$$\tau_1 \equiv c_0^H \tau = c_0^H e^{-\frac{t}{4M}}$$
The saddle point approximation

The Bogoliubov coefficients $\alpha_{\omega k}$ and $\beta_{\omega k}$ are given by

$$\alpha_{\omega k} = c(r_1) \int dt \ e^{i(S + Mt + \omega t)/l_P^2}, \quad \beta_{\omega k} = c(r_1) \int dt \ e^{i(S + Mt - \omega t)/l_P^2}. \tag{26}$$

The above integrals will be computed using the saddle point method

where $l_P^2$ plays the role of asymptotic parameter. From what we derived in the previous section, the exponent appearing in the integrands, multiplied by $-il_P^2$ apart from $q(r_1)$ which is constant in time and common to both coefficients, are respectively

$$2m^3 \tau_1 + t \tau_1^2 + m^2(s + 1) \tau_1^2 \pm \omega t \quad \text{with} \quad \tau_1 = c_0^H \tau \tag{27}$$

where we used the notation of eq. (??). For the $\alpha_{\omega k}$ case (i.e. upper sign) the saddle point is given by the value of time $t$ which satisfies

$$0 = -H(t) + M + \omega = -m\tau_1 - \frac{t}{m^2} \tau_1^2 - s\tau_1^2 + \omega \tag{28}$$
which being $\omega > 0$ has solution for real $t$ and thus at a real value of the exponent in eq.(26). On the contrary for the $\beta_{\omega k}$ case (lower sign), the saddle point equation

$$0 = -H(t) + M - \omega = -m \tau_1 - \frac{t}{m^2} \tau_1^2 - s \tau_1^2 - \omega$$

(29)

has solution for complex $t$. At such a value of time the exponent (27) (lower sign) equals

$$B = -2m^2 \omega - t(\tau_1^2 + \omega) - (s - 1)m^2 \tau_1^2.$$  

(30)

The solution of eq.(29) to second order in $\omega$, which is the order we are interested in, is given by

$$\tau_1 = -\frac{\omega}{m} \left(1 - \frac{2\omega}{m^2} \log\left(-\frac{\omega}{c_0 H m}\right) + \frac{s \omega}{m^2}\right).$$

(31)

From eq.(30) we see that to find the imaginary part of such exponent to order $\omega^2$ we simply need the imaginary part of $t$ to first order in $\omega$. Using (31) we have

$$\text{Im } t = -2\pi m^2 \left(1 - \frac{2\omega}{m^2}\right).$$

(32)
Substituting into eq.(30) we find

\[ \text{Im } B = 2\pi m^2 \omega (1 - \frac{\omega}{m^2}) = 4\pi M \omega (1 - \frac{\omega}{2M}) \]  

(33)

which according to (26) has to be divided by \( l_P^2 \). Thus we have

\[ \frac{|\beta_{\omega k}|^2}{|\alpha_{\omega k}|^2} = e^{-8\pi \frac{M \omega}{l_P^2} (1 - \frac{\omega}{2M})} \]

(34)

which is independent of \( k \).

We see from eq.(23) that for \( t \to +\infty \), \( H(t) \) tends to \( M \) and thus the time Fourier transform of the exponential of the action (25) which refers to the whole system has a singularity at the frequency \( M \).

\[ F(\omega) d\omega = \frac{1}{2\pi} \sum_k |\beta_{\omega k}|^2 d\omega \]

Using the property of the Bogoliubov coefficients

\[ \sum_k (\alpha_{\omega k} \alpha_{\omega k}^* - \beta_{\omega k} \beta_{\omega k}^*) = \delta_{\omega,\omega'} \]  

(35)
\[
F(\omega)d\omega = \frac{1}{2\pi} \frac{\sum_k |\beta_{\omega k}|^2}{\sum_{k'}(|\alpha_{\omega k'}|^2 - |\beta_{\omega k'}|^2)}d\omega
\]

one reaches for the flux of the Hawking radiation

\[
F(\omega)d\omega = \frac{1}{2\pi} \frac{1}{e^{\frac{8\pi M\omega}{l^2_p}(1-\frac{\omega}{2M})} - 1}d\omega .
\]

This completes the explicit derivation of the \(\omega^2\) correction to the Hawking formula from the time Fourier transform of the semiclassical modes.

An alternative way to derive the above result was given by Keski-Vakkuri and Kraus (Nucl.Phys. B491 (1997), arXiv:hep-th/9610045) where it is proven that for the \(\beta_{\omega k}\) coefficient the imaginary part of the action at the saddle point (29) is given by

\[
\text{Im} \int_{r_0}^{r_1} p_c dr = \text{Im} \int_{2H}^{2M} p_c dr = \frac{\pi}{2} \left( (2M)^2 - (2H)^2 \right) = 4\pi M \omega \left( 1 - \frac{\omega}{2M} \right)
\]

which is equivalent to eq.(33). The importance of equation (37) is to show directly how the “tunneling” is due only to the imaginary part of the “space part” of the action.
With regard to the validity of the expansion we see from the saddle point value (28, 29)

$$ \frac{A(e^{\frac{k}{2M}} - 1)}{\sqrt{2M}E} e^{-\frac{t}{4M}} \approx \frac{\omega}{2M} $$  \hspace{1cm} (38)

that the series if effectively an expansion in $\omega/M$ and thus expected to hold for $\omega/M \ll 1$. From eq.(38) we see that for a given $\omega$, large values of the wave number $k$ contribute at times $t$ which grow like $2k$. The typical $\omega$ for the radiation emitted by a black hole of mass $M$ is according to eq.(34)

$$ \omega \approx \frac{l_P^2}{8\pi M} $$  \hspace{1cm} (39)

and thus the approximation expected to be reliable at the typical frequency (39) or below for $l_P^2/8\pi M^2 \ll 1$ i.e. for black holes of mass of a few Planck masses or of higher mass. Higher order in the saddle point gives corrections $O(l_P^2/M^2)$. 
The relation by Keski-Vakkuri and Kraus

\[ 2\text{Im} \int_{r_0}^{r_1} p_c dr = \text{Im} \int_{2H}^{2M} p_c dr = \pi \frac{1}{2} ((2M)^2 - (2H)^2) = 8\pi M \omega \left(1 - \frac{\omega}{2M}\right) \]

(40)

Gives rise to the tunneling picture similar to ordinary Q.M. (without back reaction corrections; geodesic motion of a massless particle in the external gravitational field) (many papers).
Conclusions

1. There are rather clean derivations of the Hawking spectrum in the external field treatment but the external field treatment violates energy conservation.

2. The semiclassical approach takes into account the energy conservations and allows to compute through the convergent late-time expansion the back reaction effects to order $\omega/M$.

3. The late time expansion gives agrees with the result of Keski-Vakkuri and Kraus.

4. The derivation gives the back reaction effects and it is independent of the tunneling method: in the tunneling interpretation only the imaginary part of the “space part” of the action is relevant not the imaginary part of the whole action.
Appendix

Objections to the simple formula

1) The above is not invariant under canonical transformations: one should use
\[ \frac{1}{2} \text{Im} \oint p_c dr \]
but then factor 1/2 (Chowdhury)

2) In the name of covariance one should take into account also the “time-part” of the action (Akhmedova ...), but then the action
\[ \int_{r_i}^{r_o} (p_c dr - H dt) \]
becomes a constant which vanishes inside and thus is zero (Pizzi)

3) One should split the two contributions and take different prescriptions then
\[ \int_{r_i}^{r_o} (p dr - H dt) \]
becomes the double as wanted (Akhmedov...) cured by taking $1/2 \ast \oint$ (Singleton...)

4) One should take different prescription in treating the two contribution equivalent to neglect the time contribution (Zerbini, Vanzo ... ) but not going to the $1/2 \ast \oint$ otherwise one obtains $1/2$.

The mode analysis is \textit{completely independent} of the tunneling picture.
Units and order of magnitudes

Time, mass, energy, momentum, measured as lengths. Wave length

\[ \omega \approx \frac{l_P^2}{8\pi M} \quad \frac{\lambda}{2\pi} \approx 4\pi r_S \]

Evaporation time

\[ t = \frac{5120\pi G^2 M^3}{\hbar c^4} \]  \hspace{1cm} (41)

\( l_P = 1.62 \times 10^{-35} m \)

For the sun \( t = 5 \times 10^{74} s; \)

for \( M = 200 \, 000 \, kg, \ t = 0.67 \, s \) and \( T = 50 \, TeV; \)

for the Planck mass \((= 1.21 \times 10^{19} \, GeV/c^2 = 2.1710^{-8} \, kg)\) \( t = 8.6 \times 10^{-40} \, s. \)
The Unruh modes

\[ ds^2 = -16M^2 e^{-\frac{r^2}{4M^2}} dU dV + r^2 d\Omega^2 \]  \hspace{1cm} (42)

On \( H_- \) given by \( V = 0 \) \( \xi = \partial_U \) is a Killing vector.

\[ \mathcal{L}_\xi g_{\mu\nu} = 0. \]  \hspace{1cm} (43)

\[ U = -4M e^{-\frac{v}{4M}} \quad V = 4M e^{\frac{v}{4M}} \]  \hspace{1cm} (44)

Null coordinates

\[ u = r + 2M(\log r/2M - 1) - t \quad v = r + 2M(\log r/2M - 1) + t \]  \hspace{1cm} (45)

\[ \phi = e^{ikU} \]  \hspace{1cm} (46)
Shell dynamics

We summarize here the essential formulas of the shell dynamics. One starts from the usual Hilbert-Einstein action to which the shell action is added

\[
S = \frac{1}{16\pi G} \int R \sqrt{-g} \, d^4 x + S_{\text{shell}}.
\]

(47)

We shall in the following use \( c = G = 1 \) which simply means that masses acquire the dimension of length i.e. they are measured by the related Schwarzschild radius divided by 2. As usual in gravity it is better to work on a bounded region of space-time. Employing the general spherically symmetric metric (2) the action can be rewritten in Hamiltonian form as

\[
S = \int_{t_i}^{t_f} dt \int_{r_i}^{r_e} dr \left( \pi_L \dot{L} + \pi_R \dot{R} - N \mathcal{H}_t - N^r \mathcal{H}_r \right) + \int_{t_i}^{t_f} dt \left( -N^r \pi_L L + \frac{N R R'}{L} \right) \bigg|_{r_i}^{r_e} \\
\quad + \int_{t_i}^{t_f} dt \, \hat{p} \dot{\hat{r}}
\]

(48)

where \( \hat{r} \) denotes the radial coordinate of the shell. The constraints are
given by
\[ \mathcal{H}_r = \pi_R R' - \pi'_L L - \hat{p} \delta(r - \hat{r}), \] (49)
\[ \mathcal{H}_t = \frac{RR''}{L} + \frac{R'^2}{2L} + \frac{L\pi_L^2}{2R^2} - \frac{RR'L'}{L^2} - \frac{\pi_L \pi_R}{R} - \frac{L}{2} + \sqrt{\hat{p}^2 L^{-2} + m^2} \delta(r - \hat{r}). \] (50)

The Painlevé-Gullstrand gauge is defined by \( L \equiv 1 \). There is still one gauge freedom in the choice of \( R(r) \). In virtue of the constraints \( R'(r) \) has to be discontinuous at \( r = \hat{r} \). Here we will adopt the “outer gauge” [8] defined by \( R(r) = r \) for \( r \geq \hat{r} \) i.e. in the massless case
\[ R(r) = r + \frac{\hat{p}}{\hat{r}} g(r - \hat{r}) \] (51)
with \( g \) smooth function of support \([-l, 0]\), \( g(0) = 0 \) and \( g'(0-) = 1 \).

Other gauges could well be used The constraints can be solved and the action in the outer gauge takes the form
\[ S = \int_{t_i}^{t_f} \left( p_c \dot{r} - \dot{M}(t) \int_{r_i}^{\hat{r}(t)} \frac{\partial F}{\partial M} dr - HN(r_e) + MN(r_i) \right) dt \] (52)
where $F$ is the generating function

$$F = RW + RR'(\mathcal{L} - \mathcal{B})$$

(53)

with

$$W = \sqrt{R'^2 - 1 + \frac{2M}{R}}, \quad \mathcal{L} = \log(R' - W), \quad \mathcal{B} = \sqrt{\frac{2M}{R}} + \log \left(1 - \frac{\sqrt{2M}}{R}\right).$$

(54)

The general expression of the conjugate momentum $p_c$ is [8]

$$p_c = R(\Delta \mathcal{L} - \Delta \mathcal{B})$$

(55)

where $\Delta$ represents the discontinuity of the related quantities across the shell position $\hat{r}$. Contrary to $\hat{p}$, $p_c$ is a gauge invariant quantity within the Painlevé class of gauges. Its expression for the case of a massless shell is given by eq.(5). Normalizing the lapse function $N$, which is constant for $r > \hat{r}$, as $N(r_e) = 1$ we have from the expression (5) of $p_c$ and action (4) the equation of motion

$$\frac{\partial H}{\partial p_c} = 1 - \sqrt{\frac{2H}{\hat{r}}} = \dot{\hat{r}}$$

(56)
\[ c^0_H = \left( \frac{e^k}{m^2} - 1 \right) \frac{A}{\exp\left( - \frac{A(A+4m)}{4m^2} \right)} \]  

(57)

\[ A = \sqrt{r_1} - m \]  

(58)
References


