Can second order gravity theory explain acceleration?

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Outline

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- General relativity requires a modification at cosmological distance scales.
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- Dark energy of unknown nature is responsible for the accelerating expansion of the Universe.
- General relativity requires a modification at cosmological distance scales.
- Higher order gravity theories in vacuum derived from Lagrangians of the form, $L = f(R)$, are conformally equivalent to the Einstein field equations with a scalar field as a matter source. The scalar field is

$$\phi = \sqrt{\frac{3}{2}} \ln f'(R)$$

and the potential is given by

$$V = \frac{1}{2 (f')^2} (R f' - f).$$
A well known example: $R + \alpha R^2$ theory, with a potential function

$$V(\phi) = \frac{1}{8\alpha} \left(1 - e^{-\sqrt{2/3}\phi}\right)^2 \equiv V_\infty \left(1 - e^{-\sqrt{2/3}\phi}\right)^2$$
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If matter fields are present, the corresponding field equations are formally the Einstein equations with a scalar field coupled to matter.
Field equations for homogeneous and isotropic spacetimes

Ordinary matter is described by a perfect fluid with equation of state \( p = (\gamma - 1)\rho \), where \( 0 < \gamma \leq 2 \).
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The Friedmann equation,

\[
H^2 + \frac{k}{a^2} = \frac{1}{3} \left( \rho + \frac{1}{2} \dot{\phi}^2 + V(\phi) \right),
\]

the Raychaudhuri equation,

\[
\dot{H} = -\frac{1}{2} \dot{\phi}^2 - \frac{\gamma}{2} \rho + \frac{k}{a^2},
\]

the equation of motion of the scalar field,

\[
\ddot{\phi} + 3H \dot{\phi} + V'(\phi) = \frac{4 - 3\gamma}{\sqrt{6}} \rho,
\]

and the conservation equation,

\[
\dot{\rho} + 3\gamma \rho H = -\frac{4 - 3\gamma}{\sqrt{6}} \rho \dot{\phi}.
\]
Multi-exponenotional potentials (assisted inflation)

\[ V(\phi) = \sum_{i=1}^{N} V_i e^{-k_i \phi}, \quad V_i, k_i \in \mathbb{R} \]
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There exists a well established mathematical procedure for the investigation of scalar field cosmologies with exponential potentials in the context of dynamical systems theory. It consists in the introduction of the so-called expansion-normalized variables:

\[ x = \frac{\dot{\phi}}{\sqrt{6H}}, \quad y_i = \sqrt{\frac{V_i e^{-k_i \phi}}{3H^2}}, \quad \Omega = \frac{\rho}{3H^2}, \quad \frac{dt}{d\tau} = \frac{1}{H}. \]

The evolution equations for flat \((k = 0)\), models become

\[
\begin{align*}
x' &= x (-2 + q) + \sqrt{\frac{3}{2}} \sum_{i=1}^{N} k_i y_i^2, \\
y_i' &= y_i \left( 1 - \sqrt{\frac{3}{2}} k_i x + q \right), \\
\Omega' &= \Omega \left[ (-3\gamma + 2) + 2q \right], \\
H' &= - (1 + q) H,
\end{align*}
\]
The prime denotes differentiation with respect to the new time coordinate $\tau$ and

$$q = \frac{3\gamma - 2}{2} \Omega + 2x^2 - \sum_{i=1}^{N} y_i^2.$$

The Friedmann equation yields the constraint

$$1 = \Omega + x^2 + \sum_{i=1}^{N} y_i^2.$$
Advantages of this method:

- The evolution equation of the Hubble function $H$ decouples from the remaining equations. As a result, the dimension of the dynamical system reduces by one.
- The phase space of the system is bounded.

Limitations of the formalism:

- It can be applied only in the case of an exponential potential, due to the fact that this potential has the nice property that $V' \propto V$ which allows the introduction of expansion-normalized variables. For arbitrary potentials the reduction of the dimensionality of the dynamical system is lost.
- The transformation to the expansion-normalized variables becomes singular when $H = 0$ and consequently, it does not give a complete description of the evolution for closed models. In fact, at the time of maximum expansion the Hubble parameter becomes zero and therefore, the time coordinate $\tau$ cannot be used past the instant of maximum expansion.
Adapt the above well known formalism to the potential function

\[ V(\phi) = V_\infty \left(1 - e^{-\sqrt{2/3} \phi}\right)^2. \]

This function can be written as

\[ V(\phi) = V_\infty + V_\infty e^{-2\sqrt{2/3} \phi} - 2V_\infty e^{-\sqrt{2/3} \phi} \equiv \sum_{i=1}^{3} V_i e^{-k_i \phi}. \]

The state space consists of 6-tuples \((x, y_1, y_2, y_3, \Omega, H) \in \mathbb{R}^6\). The variables \(y_i\) are not independent, for example \(y_3^2 = -2y_1y_2\).
With this choice, the corresponding dynamical system is

\[
x' = -3x + \frac{4 - 3\gamma}{2} \Omega + xQ - 2y_1y_2 + 2y_2^2,
\]

\[
y_1' = y_1Q,
\]

\[
y_2' = y_2(-2x + Q),
\]

\[
\Omega' = \Omega[-3\gamma - (4 - 3\gamma)x + 2Q],
\]

\[
H' = -HQ,
\]

where

\[
Q = 3x^2 + \frac{3\gamma}{2} \Omega \equiv 1 + q.
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\]

The planes \(y_1 = 0\) and \(y_2 = 0\) are invariant sets of the dynamical system.
A further reduction of the dimension of the system is obtained using the constraint and we end up with a three-dimensional system. The system has 11 equilibrium points, but the physically acceptable for expanding models are fewer and they are shown in the table.

<table>
<thead>
<tr>
<th>((x, y_1, y_2))</th>
<th>Eigenvalues</th>
<th>(\Omega)</th>
<th>(\Omega_\phi)</th>
<th>(a(t))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((\pm 1, 0, 0))</td>
<td>Positive</td>
<td>0</td>
<td>1</td>
<td>(t^{1/3})</td>
</tr>
<tr>
<td>((\frac{4-3\gamma}{6-3\gamma}, 0, 0))</td>
<td>(\lambda_{1,2} &gt; 0, \lambda_3 &lt; 0) for (\gamma &lt; 5/3)</td>
<td>(\frac{4(3\gamma-5)}{9(\gamma-2)^2})</td>
<td>(1 - \Omega)</td>
<td>(t^{(6-3\gamma)/(8-3\gamma)})</td>
</tr>
<tr>
<td>((\frac{2}{3}, 0, \frac{\sqrt{5}}{3}))</td>
<td>(\lambda_{1,2} &lt; 0, \lambda_3 &gt; 0)</td>
<td>0</td>
<td>1</td>
<td>(t^{3/4})</td>
</tr>
<tr>
<td>((0, 1, 0))</td>
<td>(-3, -3\gamma, 0)</td>
<td>0</td>
<td>1</td>
<td>(\exp\left(\sqrt{\frac{V_\infty}{3}}t\right))</td>
</tr>
</tbody>
</table>
Projecting our system on the invariant set \( y_2 = 0 \) we obtain

\[
x' = 2 - \frac{3\gamma}{2} + \left( \frac{3\gamma}{2} - 3 \right) x + \left( \frac{3\gamma}{2} - 2 \right) x^2 + \left( 3 - \frac{3\gamma}{2} \right) x^3 + \left( \frac{3\gamma}{2} - 2 \right) y_1^2 - \frac{3\gamma}{2} xy_1^2
\]

\[
y_1' = y_1 \left( \frac{3\gamma}{2} + \left( 3 - \frac{3\gamma}{2} \right) x^2 - \frac{3\gamma}{2} y_1^2 \right).
\]

It is evident by the linearization theorem that the equilibrium \((0, 1)\) is a stable node, and in fact the phase portrait of the two dimensional system is future asymptotically stable.

Figure:
The center manifold theorem implies that there exists a local 2-dimensional stable manifold through \((0, 1, 0)\), i.e., all trajectories asymptotically approaching \((0, 1, 0)\) as \(t \to \infty\), lie on a 2-dimensional invariant manifold.
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Computation of the local center manifold at \((0, 1, 0)\) shows that

**Proposition**

*The equilibrium point \((0, 1, 0)\) is locally asymptotically unstable.*
Proof of Proposition

The procedure to determine the local center manifold is fairly systematic and will be accomplished in the following steps.

1. We shift the fixed point to \((0, 0, 0)\) by setting \(\tilde{y}_1 = y_1 - 1\) and write the system in vector notation as

\[
\dot{z} = Az + F(z),
\]

where \(A\) is the linear part of the vector field and \(F(0) = 0\).

2. Using the matrix \(T\) which transforms the linear part of the vector field into Jordan canonical form, we define new variables, \((z, z_1, z_2) \equiv z\), by the equations \(x = Tz\) so that the system becomes

\[
\dot{x} = T^{-1}ATx + T^{-1}F(Tx).
\]

Denoting the canonical form of \(A\) by \(B\) we finally obtain the system

\[
\dot{x} = Bx + f(x),
\]

where \(f(x) := T^{-1}F(Tx)\).
In components:

\[
\begin{bmatrix}
\dot{x} \\
\dot{u}_1 \\
\dot{u}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & 0 \\
0 & -3 & 0 \\
0 & 0 & -3\gamma
\end{bmatrix}
\begin{bmatrix}
x \\
u_1 \\
u_2
\end{bmatrix} + f(x)
\]
3. The system is written in diagonal form

\[
\begin{align*}
\dot{x} &= Cx + f(x, u) \\
\dot{u} &= Pu + g(x, u),
\end{align*}
\]

where \((x, u) \in \mathbb{R} \times \mathbb{R}^3\), \(C\) is the zero \(1 \times 1\) matrix, \(P\) is a \(2 \times 2\) matrix with negative eigenvalues and \(f, g\) vanish at \(0\) and have vanishing derivatives at \(0\). The center manifold theorem asserts that there exists a 1-dimensional invariant local center manifold \(W^c(0)\) of the system tangent to the center subspace (the \(u = 0\) space) at \(0\). Moreover, \(W^c(0)\) can be represented as

\[
W^c(0) = \{ (x, u) \in \mathbb{R} \times \mathbb{R}^3 : u = h(x), \ |x| < \delta \}; \quad h(0) = 0, \ D h(0) = 0,
\]

for \(\delta\) sufficiently small. The restriction of the system to the center manifold is

\[
\dot{x} = f(x, h(x)).
\]

According to center manifold theorem, if the origin \(x = 0\) is stable then the origin of the 3-D system is also stable. Therefore, we have to find the local center manifold, i.e., the problem reduces to the computation of \(h(x)\).
4. Substituting \( \mathbf{u} = \mathbf{h} (x) \) in the second component and using the chain rule, 
\( \dot{\mathbf{u}} = D\mathbf{h} (x) \dot{x} \), one can show that the function \( \mathbf{h} (x) \) that defines the local center manifold satisfies

\[
D\mathbf{h} (x) \left[ f (x, \mathbf{h} (x)) \right] - P\mathbf{h} (x) - g (x, \mathbf{h} (x)) = 0.
\]

This condition allows for an approximation of \( \mathbf{h} (x) \) by a Taylor series at \( x = 0 \). Since \( \mathbf{h} (0) = 0 \) and \( D\mathbf{h} (0) = 0 \), it is obvious that \( \mathbf{h} (x) \) commences with quadratic terms.
We substitute

\[ h(x) =: \begin{bmatrix} h_1(x) \\ h_2(x) \end{bmatrix} = \begin{bmatrix} a_1 x^2 + a_2 x^3 + O(x^4) \\ b_1 x^2 + b_2 x^3 + O(x^4) \end{bmatrix} \]

and set the coefficients of like powers of \( x \) equal to zero to find the unknowns \( a_1, b_1, \ldots \).
5. We find

\[ \dot{x} = \left( \frac{4}{3} + 6\gamma \right) x^2 + O(x^3). \]

It is obvious that the origin \( x = 0 \) is asymptotically unstable (saddle point) and therefore, the origin \( x = 0 \) of the full three-dimensional system is unstable. This completes the proof.
Closed models \((k = +1)\)

- The decoupling property of \(H\) is lost.
Closed models \((k = +1)\)

- The decoupling property of \(H\) is lost.
- Five-dimensional system, but much richer behaviour.