Second-order gravitational self-force

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A small extended body moving through spacetime

**Fundamental question**
- how does a body’s gravitational field affect its own motion?

**Regime: asymptotically small body**
- examine spacetime \((\mathcal{M}, g_{\mu\nu})\) containing body of mass \(m\) and external lengthscales \(\mathcal{R}\)
- seek representation of motion in limit \(\epsilon = m/\mathcal{R} \ll 1\)
Gravitational self-force

- treat body as source of perturbation of external background spacetime \((\mathcal{M}_E, g_{\mu\nu})\)
  \[
g_{\mu\nu} = g_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + \ldots
  \]
- \(h^{(n)}_{\mu\nu}\) exerts \textit{self-force} on body
- self-force at linear order in \(\epsilon\) first calculated in 1996 [Mino, Sasaki, and Tanaka], now on firm basis [Gralla & Wald; Pound; Harte]
Canonical example: extreme-mass-ratio inspiral

- solar-mass neutron star or black hole orbits supermassive black hole
- $m =$ mass of smaller body, $\mathcal{R} \sim M =$ mass of large black hole
- $(\mathcal{M}_E, g_{\mu\nu}) =$ Kerr spacetime of large black hole

Why second order?

- inspiral occurs very slowly, on timescale $1/\epsilon$
  $\Rightarrow$ need $O(\epsilon^2)$ terms in acceleration to get trajectory correct at $O(1)$
- also useful to complement PN and NR
How to determine motion: buffer region

- define buffer region by $m \ll r \ll R$
- because $m \ll r$, can treat mass as small perturbation of external background
- because $r \ll R$, can use information about small body

inner region ($r \sim m$)

buffer region

external universe ($r \sim R$)
Matched asymptotic expansions: *inner expansion*

**Zoom in on body**

- map $\psi$ keeps size of body fixed, sends other distances to infinity (e.g., using coords $\sim r/\epsilon$)
- unperturbed body defines background spacetime $g_{I\mu\nu}$ in inner expansion
- buffer region at asymptotic infinity $\Rightarrow$ can define multipole moments
Matched asymptotic expansions: *outer expansion*

Send body to zero size around a worldline

- map $\varphi$ shrinks body to zero size, holding other distances fixed
- build metric $g_{\mu\nu} + \epsilon h_{\mu\nu}^{(1)} + \epsilon^2 h_{\mu\nu}^{(2)} + \ldots$ in external universe (outside buffer region) subject to *matching condition*: in coords centered on $\gamma$, metric in buffer region must agree with inner expansion
Metric in buffer region

Expansion for small $r$

- presence of any compact body in inner region leads to
  \[ h_{\mu\nu}^{(1)} = \frac{1}{r} h_{\mu\nu}^{(1,-1)} + h_{\mu\nu}^{(1,0)} + rh_{\mu\nu}^{(1,1)} + O(r^2) \]
  \[ h_{\mu\nu}^{(2)} = \frac{1}{r^2} h_{\mu\nu}^{(2,-2)} + \frac{1}{r} h_{\mu\nu}^{(2,-1)} + h_{\mu\nu}^{(2,0)} + O(r) \]

  where $r$ is distance from $\gamma$

- most divergent terms are background spacetime in inner expansion:
  \[ g_{I\mu\nu} = \eta_{\mu\nu} + \frac{1}{r} h_{\mu\nu}^{(1,-1)} + \frac{1}{r^2} h_{\mu\nu}^{(2,-2)} + O(1/r^3) \]

Relating worldline to body

- define $\gamma$ to be worldline of body iff mass dipole terms vanish in coords centered on $\gamma$
Solving the EFE with an accelerated source

Expansion of EFE

- allow $\gamma$ to depend on $\epsilon$ and assume outer expansion of form
  \[
g_{\mu\nu}(x, \epsilon) = g_{\mu\nu}(x) + h_{\mu\nu}(x; \gamma)
  = g_{\mu\nu}(x) + \epsilon h^{(1)}_{\mu\nu}(x; \gamma) + \epsilon^2 h^{(2)}_{\mu\nu}(x; \gamma) + \ldots
  \]
- need a method of systematically solving for each $h^{(n)}_{\mu\nu}$
  $\Rightarrow$ impose Lorenz gauge on total perturbation: $\nabla_\mu \bar{h}_{\mu\nu} = 0$
- linearized Einstein tensor $\delta G_{\mu\nu}$ becomes a wave operator and EFE becomes a weakly nonlinear wave equation:
  \[
  \Box \bar{h}_{\mu\nu}[\gamma] + 2R_{\mu}^{\rho \nu \sigma} \bar{h}_{\rho\sigma}[\gamma] = 2\delta^2 G_{\mu\nu}[h] + \ldots
  \]
  (no stress-energy tensor because equation written outside body)
- can be split into wave equations for each subsequent $h^{(n)}_{\mu\nu}[\gamma]$ and exactly solved for arbitrary $\gamma$
- $\nabla_\mu \bar{h}_{\mu\nu} = 0$ determines acceleration of $\gamma$
Solving the EFE with an accelerated source

Expansion of EFE

- allow $\gamma$ to depend on $\epsilon$ and assume outer expansion of form

$$g_{\mu\nu}(x, \epsilon) = g_{\mu\nu}(x) + h_{\mu\nu}(x; \gamma)$$

$$= g_{\mu\nu}(x) + \epsilon h_{\mu\nu}^{(1)}(x; \gamma_\epsilon) + \epsilon^2 h_{\mu\nu}^{(2)}(x; \gamma_\epsilon) + \ldots$$

- need a method of systematically solving for each $h_{\mu\nu}^{(n)}$

  $\Rightarrow$ impose Lorenz gauge on total perturbation: $\nabla_{\mu} \bar{h}_{\mu\nu} = 0$

- linearized Einstein tensor $\delta G_{\mu\nu}$ becomes a wave operator and EFE becomes a weakly nonlinear wave equation:

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General solution in buffer region

**First order**
- Field naturally splits in two: \( h_{\mu\nu}^{(1)} = h_{\mu\nu}^S(1) + h_{\mu\nu}^R(1) \)
- \( h_{\mu\nu}^S(1) \sim 1/r + \ldots \) defined by mass monopole \( m \)
- \( h_{\mu\nu}^R(1) \sim r^0 + \ldots \) undetermined homogenous solution regular at \( r = 0 \)
- \( \nabla_\mu \tilde{h}_{\mu\nu} = 0 \Rightarrow \dot{m} = 0 \) and \( a_{(0)}^\mu = 0 \)

**Second order**
- Field naturally splits in two: \( h_{\mu\nu}^{(2)} = h_{\mu\nu}^S(2) + h_{\mu\nu}^R(2) \)
- \( h_{\mu\nu}^S(2) \sim 1/r^2 + 1/r + \ldots \) defined by
  1. mass correction \( \delta m \)
  2. mass dipole \( M^\mu \) (set to zero with appropriate choice of \( \gamma \))
  3. spin dipole \( S^\mu \)
- \( \nabla_\mu \tilde{h}_{\mu\nu} = 0 \Rightarrow \dot{S}^\mu = 0, \delta \dot{m} = \ldots, \) and \( a_{(1)}^\mu = \ldots \)
Matching to an inner expansion

Inner expansion
- could continue with same method to find $a^{(2)}_{\mu}$ from $h^{(3)}_{\mu\nu}$
- instead, get more information from inner expansion
- assume metric in inner expansion is Schwarzschild as tidally perturbed by external universe
- write tidally perturbed Schwarzschild metric in mass-centered coordinates

Matching
- expand inner metric in buffer region (i.e., for $r \gg m$)
- demand inner and outer expansions in buffer region are related by unique gauge transformation $x^\mu \rightarrow x^\mu + \epsilon \xi^\mu + \ldots$
- restrict gauge transformation to include no translations at $r = 0$ to ensure worldline correctly associated with center of mass
Equation of motion

Self-force

- matching procedure yields acceleration

\[ a^\mu = \frac{1}{2} (g^{\mu\nu} + u^\mu u^\nu) \left( g^\nu_\rho - h^R_\nu^\rho \right) \left( h^R_\sigma_\lambda_\rho - 2 h^R_\rho_\sigma_\lambda_\lambda \right) u^\sigma u^\lambda + O(\epsilon^3) \]

where \( a^\mu = a^{(0)}_\mu + \epsilon a^{(1)}_\mu + \epsilon^2 a^{(2)}_\mu + \ldots \)

and \( h^R_\mu_\nu = \epsilon h^{(1)}_\mu_\nu + \epsilon^2 h^{(2)}_\mu_\nu + \ldots \)

- this is geodesic equation in metric \( g_{\mu\nu} + h^R_{\mu\nu} \)

- equation for more generic body will be the same, modified only by body’s multipole moments

\[ \text{body's field } h_{\alpha\beta} \quad = \quad \text{singular field } h^S_{\alpha\beta} \quad + \quad \text{regular field } h^R_{\alpha\beta} \]
Obtaining global solution

Puncture/effective-source scheme

- define $h^P_{\mu\nu}$ as small-$r$ expansion of $h^S_{\mu\nu}$ truncated at order $r$ or higher
- define $h^R_{\mu\nu} = h_{\mu\nu} - h^P_{\mu\nu} \approx h^R_{\mu\nu}$

The point...

- $h^S_{\mu\nu}$ found in buffer region suffices to determine both $h^R_{\mu\nu}$ and global solution outside body
Obtaining global solution

\[ \delta G^{\mu\nu}[h_{\rho\sigma}] = \text{full source} \]

\[ \delta G^{\mu\nu}[h^R_{\rho\sigma}] = \text{full source} - \delta G^{\mu\nu}[h^P_{\rho\sigma}] \]

Puncture/effective-source scheme

- define \( h^P_{\mu\nu} \) as small-\( r \) expansion of \( h^S_{\mu\nu} \) truncated at order \( r \) or higher
- define \( h^R_{\mu\nu} = h_{\mu\nu} - h^P_{\mu\nu} \approx h^R_{\mu\nu} \)

The point...

- \( h^S_{\mu\nu} \) found in buffer region suffices to determine both \( h^R_{\mu\nu} \) and global solution outside body
Determining the motion of a small body

- define a worldline of an asymptotically small body, even a black hole, by comparing metric in a buffer region around body in full spacetime and in background spacetime
- determine equation of motion from consistency of Einstein’s equation

Future work

- find equation for spinning, non-spherical body
- implement puncture scheme