Elasticity and spherical symmetry

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overview

- Relativistic elasticity: general description
- Killing vectors and consequences in physics
- The energy-momentum tensor for elastic materials
- The spherically symmetric case: setting up the problem
- The EFE´s
- Finding solutions

aim and method

- to study the spherically symmetric case, in general

- the results extend previous work done by G Magli , Gen Rel Grav **25**, 441-460 (1993) : “The dynamical structure of the EFE´s for a non-rotating star”

spacetime and material space

- (M,g) is the spacetime, g being a Lorentz metric with signature (-,+,+,+)
- coordinates in M: (x^a), with $a=0,1,2,3$
- (X, γ) is the material space:

  -$\dim X=3$

  -γ is a Riemannian metric – the material metric
- coordinates in X: (y^A), with $A=1,2,3$

spacetime configuration of the material

- is specified when a mapping $\Psi: M \to X$ (a submersion) is given,
- Ψ is represented in local charts by the 3 fields $y^A = y^A\,(x^b\,)$
- then $\Psi_* : T_p\, M \to T_{\Psi(p)}\, X$ is represented by the rank 3 matrix $(y^A{}_b)_p = (\, \partial y^A / \partial x^b)_p$ – the relativistic deformation gradient

the velocity field of the matter

- since Ψ_* has maximal rank 3, its kernel is spanned at each point by a single timelike vector (which we will take as normalized to unity);
- the resulting vector field $\boldsymbol{u} = u^a \partial_a$ satisfies then (if we choose it to be future oriented)

$$y^A{}_b u^b = 0, \quad u^a u_a = -1, \quad u^0 > 0$$

- $\boldsymbol{u}$ is called the velocity field of the matter

the strain tensor K_{ab}

- we will take the following definition for the strain tensor:

$$K_{ab} = k_{ab} - u_a u_b ,$$

where

$$k_{ab} \equiv (\Psi^* \gamma)_{ab} \quad \text{and} \quad K_{ab} u^b = u_a$$

- the material space is said to be in a locally relaxed state at an event $p \in M$ if, at p, $k_{ab}=h_{ab}$, where $h_{ab} = g_{ab} + u_a u_b$; otherwise it is said to be strained,
- therefore, the strain gives a "measure" of the state of strain of the material, given by the difference between k_{ab} and h_{ab}

constitutive equation of the material

- the strain tensor determines the elastic energy stored in an infinitesimal volume element of the material (or energy per particle), hence that energy will be a scalar function of K_{ab} , named the constitutive equation of the material:

$$v=v\ (I_1,\ I_2,\ I_3)$$

with I_1, I_2, I_3 any suitable chosen set of scalar invariants associated with and characterizing K_{ab} completely

the invariants

- we will follow Magli and choose

$$I_1 = \tfrac{1}{2}\,(\mathrm{Tr}\,K - 4)$$

$$I_2 = \tfrac{1}{4}\,[\mathrm{Tr}\,K^2 - (\mathrm{Tr}\,K)^2\,] + 3$$

$$I_3 = \tfrac{1}{2}\,(\det K - 1)$$

the energy density ρ

- $\rho = \varepsilon v (I_1, I_2, I_3)$

where ε is the particle number density, or

$\rho = \varepsilon_0 (\det K)^{1/2} v (I_1, I_2, I_3)$

with ε_0 the particle number density as measured in the material space

- one might show that ε and ε_0 are the particle number densities as measured with respect to the volume forms associated to h_{ab} and k_{ab}, respectively

the energy-momentum tensor T_{ab}

Any symmetric, second order covariant tensor field may be decomposed with respect to a timelike unit vector field ***v*** as:

$$T_{ab}=\rho\, v_a\, v_b+ph_{ab}+\Pi_{ab}+v_a q_b+q_a v_b$$

with

$$h_{ab} = g_{ab} + v_a\, v_b \qquad \Pi_{ab}= h_a{}^m\, h_b{}^n\,(T_{mn} -3ph_{mn})$$

$$q_a =-(T_{ab}\, v^b + \rho\, v_a) \qquad \rho = T_{ab}\, v^a\, v^b$$

$$p= 1/3\; h^{ab} T_{ab}$$

It follows that

$$h_{ab}\, v^b =0 \qquad \Pi_{ab}\, v^b = g^{ab}\, \Pi_{ab} =0 \qquad q^a\, v_a = 0.$$

the energy-momentum tensor for elastic matter

Decomposing the energy-momentum tensor T_{ab} with respect to the velocity field of the matter **u**, one has

$$T_{ab} = \rho\, u_a\, u_b + p\, h_{ab} + \Pi_{ab}$$

Thus, for elastic matter

- $q^a = 0$
- T_{ab} is of the diagonal Segre type {1, 111} or any of its degeneracies, with ***u*** its (unit) timelike eigenvector.

using an orthonormal tetrad

This means that an orthonormal tetrad exists

$$\{u_a, x_a, y_a, z_a\}$$

with

- $u_a u^a = -1, \;\; x^a x_a = y^a y_a = z^a z_a = +1$
- the mixed products equal to zero

with respect to which the energy-momentum tensor may be written as

$$T_{ab} = \rho u_a u_b + p_1 x_a x_b + p_2 y_a y_b + p_3 z_a z_b$$

with

$$p = 1/3 (p_1 + p_2 + p_3) , \qquad h_{ab} = x_a x_b + y_a y_b + z_a z_b, \text{ etc}$$

KV´s and elastic matter

If ξ is a KV and T_{ab} represents elastic matter, then

$$\mathcal{L}_{\vec{\xi}}\rho = \mathcal{L}_{\vec{\xi}}u_a = \mathcal{L}_{\vec{\xi}}h_{ab} = \mathcal{L}_{\vec{\xi}}\Pi_{ab} = \mathcal{L}_{\vec{\xi}}p = 0.$$

Therefore density, matter 4-velocity, projection tensor, anisotropic tensor and pressure all stay invariant along Killing vectors.

Also, if ***v*** is a non-degenerate unit eigenvector of T_{ab} , with eigenvalue λ , then

$$\mathcal{L}_{\vec{\xi}}\,\lambda = \mathcal{L}_{\vec{\xi}}\,v^a = 0$$

comparison with a general T_{ab}

- for a general energy-momentum tensor such that $q_a \neq 0$, if one assumes that the Lie derivative of u_a with respect to $\boldsymbol{\xi}$ vanishes, then the Lie derivative of q_a with respect to $\boldsymbol{\xi}$ also vanishes. But is this case the assumprion has to be made, since $\boldsymbol{u}$ is no longer an eigenvector of the energy-momentum tensor.

homothetic vector fields and elastic matter

Similar conclusions (although not the same) can be drawn when ξ is a proper homothetic vector field

$$\mathcal{L}_{\vec{\xi}}\, g_{ab} = 2k g_{ab} \qquad\qquad \mathcal{L}_{\vec{\xi}}\, T_{ab} = 0$$

with $k \neq 0$, in which case one has

$$\mathcal{L}_{\vec{\xi}}\, u_a = -k u_a, \qquad \mathcal{L}_{\vec{\xi}}\, \rho = k\rho, \qquad \ldots$$

spherically symmetry

Coordinates *t, r*, θ , ϕ , exist such that the metric can be written as

$$ds^2 = -a(r,t)dt^2 + b(r,t)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

This metric admits the following KV´s:

$$\vec{\xi_1} = -\cos\phi\,\partial_\theta + \cot\theta\sin\phi\,\partial_\phi$$
$$\vec{\xi_2} = \partial_\phi$$
$$\vec{\xi_3} = -\sin\phi\,\partial_\theta - \cot\theta\cos\phi\,\partial_\phi$$

They generate the 3-dimensional Lie algebra so(3).

spherical symmetry: consequences

- Any timelike vector field that stays invariant along the three KV´s can be written as

$$\vec{v} = v^t(t,r)\ \partial_t + v^r(t,r)\ \partial_r .$$

There exists a coordinate transformation taking *(t, r)* into *(t´, r´)* such that $\vec{v} = v^{t'}(t',r')\ \partial_{t'}$

- Any symmetric, second order tensor Π_{ab} such that $\Pi_{ab}v^b = 0$, $g^{ab}\ \Pi_{ab} = 0$ and with zero Lie derivative with respect to those Killing vectors, is proportional to the shear tensor σ_{ab} of ***v*** (provided it is non-zero)

consequences

- For the class os space times under consideration (elastic and spherically symmetric), the previous 2 comments apply for ***u*** (the velocity field of the matter) and Π_{ab} (the anisotropic pressure tensor) since

(i) ***u*** is the unique timelike eigenvector of T_{ab} invariant under so(3)

(ii) Π_{ab} is invariant under so(3), traceless and orthogonal to ***u***; therefore, whenever the shear of ***u*** is non-zero,

$$\Pi_{ab} = 2\lambda\sigma_{ab},$$

with

$$\sigma_{ab} = u_{(a;b)} + u_{(a}\, u_{b)} - 1/3\,\theta\, h_{ab},\ \ \theta = u^{a}{}_{;a}\,,\ \ \lambda = \lambda(t,r)$$

the 2 cases arising

- Non-shearfree – it is always possible to formally treat the elastic material as a viscous fluid with zero heat flow; the pressure anisotropy may then be explained as an effect of the viscosity

- Shearfree – the pressure anisotropy could only be a consequence of the elastic properties of the material

assumptions to build up models

- we demand that the mapping

$$\Psi: M \longrightarrow X$$

preserves the KV´s, i.e.

$$\Psi_{\star}(\xi_A) = \eta_A \qquad (A=1,2,3)$$

are KV´s for the material metric $\bar{\gamma}$ defined on the material space X.

This implies that the metric $\bar{\gamma}$ is also spherically symmetric and therefore coordinates

$$y^A = (y, \tilde{\theta}, \tilde{\phi})$$

exist with

$$y = y(t, r), \quad \tilde{\theta} = \theta, \quad \tilde{\phi} = \phi$$

and are such that $\vec{\eta}_A = \vec{\xi}_A$ are KVs of the metric $\bar{\gamma}$ and therefore, the line elements of $\bar{g}$ and $\bar{\gamma}$ may be written as:

spacetime metric and material metric

$$d\bar{s}^2 = -\bar{a}(t,r)dt^2 + \bar{b}(t,r)dr^2 + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2$$

$$d\bar{\Sigma}^2 = f^2(y)(dy^2 + y^2 d\theta^2 + y^2 sin^2\theta d\phi^2),$$

relationship with a flat material metric

The situation corresponding to *f(y)=1* (i.e. flat material metric) was studied by Magli.

The relationship between his material metric and the material metric just obtained is

$$\bar{\gamma}_{AB} = f^2(y)\gamma_{AB}$$

Applying the methodology of Magli´s paper to this nonflat material metric one can generalize his results and explore the consequences of having the function *f(y)*.

procedure

- Calculate the pulled back material metric $\bar{k} = \Psi^\star(\bar{\gamma})$
- Determine the matter velocity field $\boldsymbol{u}$
- Obtain the projection tensor $h_{ab} = u_a u_b + g_{ab}$
- Obtain the eigenvalues and invariants of the strain tensor K
- Obtain the energy-momentum tensor
- Use the contracted Bianchi identities
- Obtain a constitutive equation

eigenvalues of the strain tensor

- Eigenvalues of K: one is equal to 1 and the others are

$$\bar{s} = f^2(y)\, s = f^2(y)\, \frac{y^2}{r^2}$$

$$\bar{\eta} = f^2(y)\, \frac{\gamma^2}{\bar{\gamma}^2}\, \frac{b}{\bar{b}}\, \eta = f^2(y)\, \frac{y'^2}{\bar{\gamma}^2\, \bar{b}}$$

where $\bar{s}$ is an eigenvalue of algebraic multiplicity two, $s = \dfrac{y^2}{r^2}$ and $\eta = \dfrac{y'^2}{\gamma^2 b}$.

Invariants of $\bar{K}$:

$$\bar{I}_1 = \frac{1}{2}\left(Tr\bar{K} - 4\right)$$

$$\bar{I}_2 = \frac{1}{4}\left[Tr\bar{K}^2 - \left(Tr\bar{K}\right)^2\right] + 3$$

$$\bar{I}_3 = \frac{1}{2}\left(det\bar{K} - 1\right)$$

where

$$Tr\,\bar{K} = \bar{\eta} + 2\,\bar{s} + 1$$

$$Tr\,\bar{K}^2 = \bar{\eta}^2 + 2\,\bar{s}^2 + 1$$

$$det\,\bar{K} = \bar{\eta}\,\bar{s}^2.$$

□□□Relations between the invariants

$$\bar{I}_1 = f^2 I_1 + \frac{3}{2}\left(f^2 - 1\right) + \frac{1}{2} f^2 \eta \left(\frac{\gamma^2}{\bar{\gamma}^2}\frac{b}{\bar{b}} - 1\right)$$

etc

- Write

Expression for the energy-momentum tensor

$$\bar{T}^a_b = \bar{\rho}\,\delta^a_b - \frac{\partial\bar{\rho}}{\partial\bar{I}_3}\,det\bar{K}\,\bar{h}^a_b + \left(Tr\bar{K}\,\frac{\partial\bar{\rho}}{\partial\bar{I}_2} - \frac{\partial\bar{\rho}}{\partial\bar{I}_1}\right)\bar{k}^a_b - \frac{\partial\bar{\rho}}{\partial\bar{I}_2}\,\bar{k}^a_c\,\bar{k}^c_b$$

Components of the energy-momentum tensor

$$\bar{\mathcal{T}}_0^0 = \bar{\rho} + \frac{\dot{y}^2}{\bar{a}} \overline{\sum},$$

$$\bar{\mathcal{T}}_0^1 = -\frac{\dot{y}y'}{\bar{b}} \overline{\sum},$$

$$\bar{\mathcal{T}}_1^0 = -\frac{\bar{b}}{\bar{a}} \bar{\mathcal{T}}_0^1,$$

$$\bar{\mathcal{T}}_1^1 = \bar{\rho} - \frac{y'^2}{\bar{b}} \overline{\sum},$$

and

$$\bar{T}_2^2 = \bar{T}_3^3 = \bar{\rho} - \frac{y^2}{r^2}\left[\overline{\sum} + \left(\frac{\partial\bar{\rho}}{\partial\bar{I}_2} - f^2(y)\,\frac{y^2}{r^2}\,\frac{\partial\bar{\rho}}{\partial\bar{I}_3}\right)\left(f^4(y)\,\frac{y^2}{r^2} - f^4(y)\,\frac{y'^2}{\bar{\gamma}^2\bar{b}}\right)\right],$$

where

$$\overline{\sum} = f^2(y)\left[\frac{\partial\bar{\rho}}{\partial\bar{I}_1} - \frac{\partial\bar{\rho}}{\partial\bar{I}_2}\left(1 + 2\,f^2(y)\,\frac{y^2}{r^2}\right) + \frac{\partial\bar{\rho}}{\partial\bar{I}_3}\,f^4(y)\,\frac{y^4}{r^4}\right].$$

Relation between energy-momentum tensors

$$\bar{T}^a_b = f^3 \frac{\bar{v}}{v} \sqrt{\frac{\gamma^2 b}{\bar{\gamma}^2 \bar{b}}} \left\{ T^a_b + (1 - f^2) \frac{\partial \rho}{\partial I_3} det\bar{K} \; k^a_b \right\}$$

(relation between the energy-momentum tensors for two conformally related material metrics)

the EFE´s

The Einstein field equations, as

$$\bar{G}^a_b = 8\pi \bar{\mathcal{T}}^a_b$$

become:

$\bar{G}^0_0 = 8\pi \bar{\mathcal{T}}^0_0$:

$$-\frac{\bar{b}'}{r\bar{b}^2} - \frac{1}{r^2}\left(1 - \frac{1}{\bar{b}}\right) = \bar{\epsilon}\left[\bar{v} + \left(\bar{v} + 2\bar{\eta}\frac{\partial \bar{v}}{\partial \bar{\eta}}\right)\bar{\gamma}^2\bar{\omega}^2\right] 8\pi,$$

$\bar{G}^1_0 = 8\pi \bar{\mathcal{T}}^1_0$:

$$\frac{\dot{\bar{b}}}{r\bar{b}^2} = -\bar{\epsilon}\sqrt{\frac{\bar{a}}{\bar{b}}}\left(\bar{v} + 2\bar{\eta}\frac{\partial \bar{v}}{\partial \bar{\eta}}\right)\bar{\gamma}^2\bar{\omega}\, 8\,\pi,$$

$\bar{G}^1_1 = 8\pi \bar{\mathcal{T}}^1_1$:

$$\frac{\bar{a}'}{r\bar{a}\bar{b}} - \frac{1}{r^2}\left(1 - \frac{1}{\bar{b}}\right) = -\bar{\epsilon}\left[\bar{\gamma}^2\left(\bar{v} + 2\bar{\eta}\frac{\partial \bar{v}}{\partial \bar{\eta}}\right) - \bar{v}\right] 8\pi,$$

$\bar{G}^2_2 = 8\pi \bar{\mathcal{T}}^2_2$:

$$\frac{1}{2\bar{b}}\left[\frac{\bar{a}''}{\bar{a}} - \frac{\bar{a}'^2}{2\bar{a}^2} + \frac{1}{r}\left(\frac{\bar{a}'}{\bar{a}} - \frac{\bar{b}'}{\bar{b}}\right) - \frac{\bar{a}'\bar{b}'}{2\bar{a}\bar{b}}\right] - \frac{1}{2\bar{a}}\left(\frac{\ddot{\bar{b}}}{\bar{b}} - \frac{\dot{\bar{b}}^2}{2\bar{b}^2} - \frac{\dot{\bar{a}}\dot{\bar{b}}}{2\bar{a}\bar{b}}\right) = -\overline{\epsilon s}\frac{\partial \bar{v}}{\partial \bar{s}}\, 8\,\pi.$$

Unknowns

- The metric $\bar{a}, \bar{b}$
- The conformal factor, f, determinig the material metric γ
- The material radius, y
- The constitutive function, $\bar{v}$

what we mean by solving the problem

- Finding a metric *g* and a material metric γ of the forms written previously, such that

(i) *g* is a solution of the EFE´s for a diagonal Segre type energy-momentum tensor, having ***u*** as its timelike eigenvector and eigenvalues ρ, p_1, p_2, p_3

(ii) making sure that the DEC are satisfied

(iii) finding *f(r)* characterizing the material metric

(iv) finding functions ε, $v=v(\eta,s)$ such that $\rho = \varepsilon v$, $p_1=-2\varepsilon\eta\partial v/\partial\eta$, $p_2 = -\varepsilon s\partial v/\partial s$

main references

- Magli G and Kijowski J (1992) *Gen Rel Grav* **24**, 139
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- Brito I, Carot J, Vaz EGLR (2009) submitted
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- Tke the operator $\bar{K}^a_b = \bar{g}^{ac}\bar{h}_{cb} - \bar{u}^a\bar{u}_b$, used to measure the state of strain of the material. Then consider its eigenvalues and invariants, as follows

Special case

So far it was not clear whether one specific spacetime metric $\bar{g}$ can be associated with different conformally related material metrics. However if one assumes that $g_{ab} = \bar{g}_{ab}$, these metrics being associated, respectively, with a flat and a non flat material metric which are conformally related, then the following relationship for constitutive equations must hold:

$$\bar{v} = \frac{1}{f^3} v.$$

Why?

This is a consequence of the following relationship between energy-momentum tensors

$$\bar{\mathcal{T}}^a_b = f^3 \frac{\bar{v}}{v} \mathcal{T}^a_b$$

together with the assumption on equality of metric tensor which leads to equal energy-momentum tensors.

In this special case one also has

$$\bar{\rho} = \rho.$$