Elasticity and spherical symmetry

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overview

• Relativistic elasticity: general description
• Killing vectors and consequences in physics
• The energy-momentum tensor for elastic materials
• The spherically symmetric case: setting up the problem
• The EFE’s
• Finding solutions
aim and method

• to study the spherically symmetric case, in general

• the results extend previous work done by G Magli, Gen Rel Grav 25, 441-460 (1993) : “The dynamical structure of the EFE’s for a non-rotating star”
spacetime and material space

- $(M,g)$ is the spacetime, $g$ being a Lorentz metric with signature $(-,+,+,+)$
- coordinates in $M$: $(x^a)$, with $a=0,1,2,3$
- $(X, \gamma)$ is the material space:
  - $\dim X = 3$
  - $\gamma$ is a Riemannian metric – the material metric
- coordinates in $X$: $(y^A)$, with $A=1,2,3$
spacetime configuration of the material

- is specified when a mapping $\Psi: M \rightarrow X$ (a submersion) is given,
- $\Psi$ is represented in local charts by the 3 fields $y^A = y^A(x^b)$
- then $\Psi^*_*: T_p M \rightarrow T_{\Psi(p)} X$ is represented by the rank 3 matrix $(y^A)_b^p = (\frac{\partial y^A}{\partial x^b})_p$ – the relativistic deformation gradient
the velocity field of the matter

- since $\Psi^*$ has maximal rank 3, its kernel is spanned at each point by a single timelike vector (which we will take as normalized to unity);

- the resulting vector field $u = u^a \partial_a$ satisfies then (if we choose it to be future oriented)
  $$y^A_b u^b = 0, \quad u^a u_a = -1, \quad u^0 > 0$$

- $u$ is called the velocity field of the matter
the strain tensor $K_{ab}$

- we will take the following definition for the strain tensor:
  \[ K_{ab} = k_{ab} - u_a u_b, \]
  where
  \[ k_{ab} \equiv (\Psi^* \gamma)_{ab} \text{ and } K_{ab} u^b = u_a \]

- the material space is said to be in a locally relaxed state at an event $p \in M$ if, at $p$, $k_{ab} = h_{ab}$, where $h_{ab} = g_{ab} + u_a u_b$; otherwise it is said to be strained,

- therefore, the strain gives a “measure” of the state of strain of the material, given by the difference between $k_{ab}$ and $h_{ab}$
constitutive equation of the material

- the strain tensor determines the elastic energy stored in an infinitesimal volume element of the material (or energy per particle), hence that energy will be a scalar function of $K_{ab}$, named the constitutive equation of the material:

$$v = v \left( I_1, I_2, I_3 \right)$$

with $I_1$, $I_2$, $I_3$ any suitable chosen set of scalar invariants associated with and characterizing $K_{ab}$ completely
the invariants

- we will follow Magli and choose
  \[ I_1 = \frac{1}{2} \left( \text{Tr } K - 4 \right) \]
  \[ I_2 = \frac{1}{4} \left[ \text{Tr } K^2 - \left( \text{Tr } K \right)^2 \right] + 3 \]
  \[ I_3 = \frac{1}{2} \left( \text{det } K - 1 \right) \]
the energy density $\rho$

- $\rho = \varepsilon \nu (l_1, l_2, l_3)$

where $\varepsilon$ is the particle number density, or

$\rho = \varepsilon_0 \sqrt{\text{det} \, K} \nu (l_1, l_2, l_3)$

with $\varepsilon_0$ the particle number density as measured in the material space.

- one might show that $\varepsilon$ and $\varepsilon_0$ are the particle number densities as measured with respect to the volume forms associated to $h_{ab}$ and $k_{ab}$, respectively.
Any symmetric, second order covariant tensor field may be decomposed with respect to a timelike unit vector field $v$ as:

$$T_{ab} = \rho \, v_a \, v_b + p h_{ab} + \Pi_{ab} + v_a \, q_b + q_a \, v_b$$

with

$$h_{ab} = g_{ab} + v_a \, v_b \quad \Pi_{ab} = h_a^m \, h_b^n \, (T_{mn} - 3p h_{mn}) \quad q_a = -(T_{ab} \, v^b + \rho \, v_a) \quad \rho = T_{ab} \, v^a \, v^b$$

$$p = \frac{1}{3} \, h^{ab} T_{ab}$$

It follows that

$$h_{ab} \, v^b = 0 \quad \Pi_{ab} \, v^b = g^{ab} \quad \Pi_{ab} = 0 \quad q^a \, v_a = 0.$$
Decomposing the energy-momentum tensor $T_{ab}$ with respect to the velocity field of the matter $u$, one has

$$T_{ab} = \rho u_a u_b + p h_{ab} + \Pi_{ab}$$

Thus, for elastic matter

- $q^a = 0$
- $T_{ab}$ is of the diagonal Segre type $\{1, 111\}$ or any of its degeneracies, with $u$ its (unit) timelike eigenvector.
using an orthonormal tetrad

This means that an orthonormal tetrad exists

\[ \{ u_a, x_a, y_a, z_a \} \]

with

- \( u_a \ u^a = -1 \), \( x^a \ x_a = y^a \ y_a = z^a \ z_a = +1 \)
- the mixed products equal to zero

with respect to which the energy-momentum tensor may be written as

\[ T_{ab} = \rho \ u_a \ u_b + p_1 \ x_a \ x_b + p_2 \ y_a \ y_b + p_3 \ z_a \ z_b \]

with

\[ p = \frac{1}{3} \ (p_1 + p_2 + p_3) \]

\[ h_{ab} = x_a \ x_b + y_a \ y_b + z_a \ z_b, \text{ etc} \]
KV’s and elastic matter

If $\xi$ is a KV and $T_{ab}$ represents elastic matter, then

$$\mathcal{L}_{\xi} \rho = \mathcal{L}_{\xi} u^a = \mathcal{L}_{\xi} h_{ab} = \mathcal{L}_{\xi} \Pi_{ab} = \mathcal{L}_{\xi} p = 0.$$  

Therefore density, matter 4-velocity, projection tensor, anisotropic tensor and pressure all stay invariant along Killing vectors.

Also, if $v$ is a non-degenerate unit eigenvector of $T_{ab}$, with eigenvalue $\lambda$, then

$$\mathcal{L}_{\xi} \lambda = \mathcal{L}_{\xi} v^a = 0.$$
comparison with a general $T_{ab}$

• for a general energy-momentum tensor such that $q_a \neq 0$, if one assumes that the Lie derivative of $u_a$ with respect to $\xi$ vanishes, then the Lie derivative of $q_a$ with respect to $\xi$ also vanishes. But is this case the assumption has to be made, since $u$ is no longer an eigenvector of the energy-momentum tensor.
homothetic vector fields and elastic matter

Similar conclusions (although not the same) can be drawn when $\xi$ is a proper homothetic vector field

$$\mathcal{L}_\xi g_{ab} = 2k g_{ab} \quad \mathcal{L}_\xi T_{ab} = 0$$

with $k \neq 0$, in which case one has

$$\mathcal{L}_\xi u_a = -k u_a, \quad \mathcal{L}_\xi \rho = k \rho, \quad ...$$
spherically symmetry

Coordinates $t$, $r$, $\theta$, $\phi$, exist such that the metric can be written as

$$ds^2 = -a(r, t) dt^2 + b(r, t) dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2$$

This metric admits the following KV`s:

$$\vec{\xi}_1 = -\cos \phi \partial_\theta + \cot \theta \sin \phi \partial_\phi$$
$$\vec{\xi}_2 = \partial_\phi$$
$$\vec{\xi}_3 = -\sin \phi \partial_\theta - \cot \theta \cos \phi \partial_\phi$$

They generate the 3-dimensional Lie algebra so(3).
spherical symmetry: consequences

• Any timelike vector field that stays invariant along the three KV’s can be written as

\[ \vec{v} = v^t(t, r) \partial_t + v^r(t, r) \partial_r. \]

There exists a coordinate transformation taking \((t, r)\) into \((t', r')\) such that \(\vec{v} = v^{t'}(t', r') \partial_{t'}\).

• Any symmetric, second order tensor \(\Pi_{ab}\) such that \(\Pi_{ab} v^b = 0\), \(g^{ab} \Pi_{ab} = 0\) and with zero Lie derivative with respect to those Killing vectors, is proportional to the shear tensor \(\sigma_{ab}\) of \(v\) (provided it is non-zero)
consequences

- For the class of space times under consideration (elastic and spherically symmetric), the previous 2 comments apply for $u$ (the velocity field of the matter) and $\Pi_{ab}$ (the anisotropic pressure tensor) since

  1. $u$ is the unique timelike eigenvector of $T_{ab}$ invariant under $so(3)$

  2. $\Pi_{ab}$ is invariant under $so(3)$, traceless and orthogonal to $u$; therefore, whenever the shear of $u$ is non-zero,

$$\Pi_{ab} = 2\lambda \sigma_{ab},$$

with

$$\sigma_{ab} = u_{(a;b)} + u_{(a} u_{b)} - 1/3 \theta h_{ab}, \quad \theta = u^a ;_a, \quad \lambda = \lambda(t,r)$$
the 2 cases arising

- **Non-shearfree** – it is always possible to formally treat the elastic material as a viscous fluid with zero heat flow; the pressure anisotropy may then be explained as an effect of the viscosity

- **Shearfree** – the pressure anisotropy could only be a consequence of the elastic properties of the material
assumptions to build up models

- we demand that the mapping

\[ \Psi: M \rightarrow X \]

preserves the KV´s, i.e.

\[ \Psi_\star(\xi_A) = \eta_A \quad (A=1,2,3) \]

are KV´s for the material metric \( \bar{\gamma} \) defined on the material space \( X \).
This implies that the metric $\tilde{\gamma}$ is also spherically symmetric and therefore coordinates

$$y^A = (y, \tilde{\theta}, \tilde{\phi})$$

exist with

$$y = y(t, r), \quad \tilde{\theta} = \theta, \quad \tilde{\phi} = \phi$$

and are such that $\tilde{\eta}_A = \tilde{\xi}_A$ are KVs of the metric $\tilde{\gamma}$ and therefore, the line elements of $\tilde{g}$ and $\tilde{\gamma}$ may be written as:
spacetime metric and material metric

\[ d\bar{s}^2 = -\bar{a}(t, r)dt^2 + \bar{b}(t, r)dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \]

\[ d\Sigma^2 = f^2(y)(dy^2 + y^2 d\theta^2 + y^2 \sin^2 \theta d\phi^2), \]
The situation corresponding to $f(y)=1$ (i.e. flat material metric) was studied by Magli.

The relationship between his material metric and the material metric just obtained is

$$\bar{\gamma}_{AB} = f^2(y)\gamma_{AB}$$

Applying the methodology of Magli’s paper to this nonflat material metric one can generalize his results and explore the consequences of having the function $f(y)$. 
procedure

• Calculate the pulled back material metric $\bar{k} = \Psi^*(\bar{\gamma})$
• Determine the matter velocity field $u$
• Obtain the projection tensor $h_{ab} = u_a u_b + g_{ab}$
• Obtain the eigenvalues and invariants of the strain tensor $K$
• Obtain the energy-momentum tensor
• Use the contracted Bianchi identities
• Obtain a constitutive equation
eigenvalues of the strain tensor

- Eigenvalues of $K$: one is equal to 1 and the others are

$$\bar{s} = f^2(y) s = f^2(y) \frac{y^2}{r^2}$$

$$\bar{\eta} = f^2(y) \frac{\gamma^2}{\bar{b}} \frac{b}{\bar{b}} \eta = f^2(y) \frac{y^\prime 2}{\gamma^2 \bar{b}}$$

where $\bar{s}$ is an eigenvalue of algebraic multiplicity two, $s = \frac{y^2}{r^2}$ and $\eta = \frac{y^\prime 2}{\gamma^2 b}$.
Invariants of $\bar{K}$:

$$\bar{I}_1 = \frac{1}{2} (Tr \bar{K} - 4)$$

$$\bar{I}_2 = \frac{1}{4} \left[ Tr \bar{K}^2 - (Tr \bar{K})^2 \right] + 3$$

$$\bar{I}_3 = \frac{1}{2} (det \bar{K} - 1)$$

where

$$Tr \bar{K} = \bar{\eta} + 2 \bar{s} + 1$$

$$Tr \bar{K}^2 = \bar{\eta}^2 + 2 \bar{s}^2 + 1$$

$$det \bar{K} = \bar{\eta} \bar{s}^2.$$
Relations between the invariants

\[ \bar{I}_1 = f^2 I_1 + \frac{3}{2} (f^2 - 1) + \frac{1}{2} f^2 \eta \left( \frac{\gamma^2 b}{\gamma^2 b} - 1 \right) \]

e tc

• Write
Expression for the energy-momentum tensor

\[ \bar{T}^a_b = \bar{\rho} \delta^a_b - \frac{\partial \bar{\rho}}{\partial \bar{I}_3} \det \bar{K} \bar{h}^a_b + \left( Tr \bar{K} \frac{\partial \bar{\rho}}{\partial \bar{I}_2} - \frac{\partial \bar{\rho}}{\partial \bar{I}_1} \right) \bar{k}^a_b - \frac{\partial \bar{\rho}}{\partial \bar{I}_2} \bar{k}^a_c \bar{k}^c_b \]
Components of the energy-momentum tensor

\[ \bar{T}_0^0 = \bar{\rho} + \frac{\dot{y}^2}{\bar{a}} \bar{\sum}, \]

\[ \bar{T}_0^1 = -\frac{\dot{y} y'}{\bar{b}} \bar{\sum}, \]

\[ \bar{T}_1^0 = -\frac{\bar{b}}{\bar{a}} \bar{T}_0^1, \]

\[ \bar{T}_1^1 = \bar{\rho} - \frac{y'^2}{\bar{b}} \bar{\sum}, \]
\begin{align*}
\bar{T}_2^2 &= \bar{T}_3^3 = \bar{\rho} \frac{y^2}{r^2} \left[ \sum \right. + \left( \frac{\partial \bar{\rho}}{\partial \bar{I}_2} - f^2(y) \frac{y^2}{r^2} \frac{\partial \bar{\rho}}{\partial \bar{I}_3} \right) \left( f^4(y) \frac{y^2}{r^2} - f^4(y) \frac{y'^2}{\gamma^2 b} \right) \right],
\end{align*}

where

\begin{align*}
\sum &= f^2(y) \left[ \frac{\partial \bar{\rho}}{\partial \bar{I}_1} - \frac{\partial \bar{\rho}}{\partial \bar{I}_2} \left( 1 + 2 f^2(y) \frac{y^2}{r^2} \right) + \frac{\partial \bar{\rho}}{\partial \bar{I}_3} f^4(y) \frac{y^4}{r^4} \right].
\end{align*}
Relation between energy-momentum tensors

\[ \bar{T}^a_b = f^3 \frac{\bar{v}}{v} \sqrt{\frac{\gamma^2_b}{\gamma^2_b}} \left\{ T^a_b + (1 - f^2) \frac{\partial \rho}{\partial I_3} \det \bar{K} \ k^a_b \right\} \]

(relation between the energy-momentum tensors for two conformally related material metrics)
The Einstein field equations, as

\[ \bar{G}^a_b = 8\pi \bar{T}^a_b \]

become:
\[ \bar{G}_0^0 = 8\pi \bar{T}_0^0: \]
\[ -\frac{\bar{b}'}{r\bar{b}^2} - \frac{1}{r^2} \left( 1 - \frac{1}{\bar{b}} \right) = \bar{\varepsilon} \left[ \bar{v} + \left( \bar{v} + 2\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} \right) \bar{\gamma}^2 \bar{\omega}^2 \right] 8\pi, \]

\[ \bar{G}_0^1 = 8\pi \bar{T}_0^1: \]
\[ \frac{\bar{b}'}{r\bar{b}^2} = -\bar{\varepsilon} \sqrt{\frac{\bar{a}}{\bar{b}}} \left( \bar{v} + 2\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} \right) \bar{\gamma}^2 \bar{\omega} 8\pi, \]

\[ \bar{G}_1^1 = 8\pi \bar{T}_1^1: \]
\[ \frac{\bar{a}'}{r\bar{a}\bar{b}} - \frac{1}{r^2} \left( 1 - \frac{1}{\bar{b}} \right) = -\bar{\varepsilon} \left[ \bar{\gamma}^2 \left( \bar{v} + 2\bar{\eta} \frac{\partial \bar{v}}{\partial \bar{\eta}} \right) - \bar{v} \right] 8\pi, \]

\[ \bar{G}_2^2 = 8\pi \bar{T}_2^2: \]
\[ \frac{1}{2\bar{b}} \left[ \frac{\bar{a}''}{\bar{a}} - \frac{\bar{a}'^2}{2\bar{a}^2} + \frac{1}{r} \left( \frac{\bar{a}'}{\bar{a}} - \frac{\bar{b}'}{\bar{b}} \right) - \frac{\bar{a}''_b}{2\bar{a}\bar{b}} \right] - \frac{1}{2\bar{a}} \left( \frac{\bar{b}'}{\bar{b}} - \frac{\bar{b}^2}{2\bar{b}^2} - \frac{\bar{a}''\bar{b}}{2\bar{a}\bar{b}} \right) = -\bar{c}_s \frac{\partial \bar{v}}{\partial \bar{s}} 8\pi. \]
Unknowns

- The metric $\bar{a}, \bar{b}$
- The conformal factor, $f$, determining the material metric $\gamma$
- The material radius, $y$
- The constitutive function, $\bar{v}$
what we mean by solving the problem

• Finding a metric $g$ and a material metric $\gamma$ of the forms written previously, such that
  
  (i) $g$ is a solution of the EFE’s for a diagonal Segre type energy-momentum tensor, having $u$ as its timelike eigenvector and eigenvalues $\rho, p_1, p_2, p_3$
  
  (ii) making sure that the DEC are satisfied
  
  (iii) finding $f(r)$ characterizing the material metric
  
  (iv) finding functions $\epsilon, \nu=\nu(\eta, s)$ such that $\rho = \epsilon \nu, \rho_1 = -2\epsilon \eta \partial \nu / \partial \eta, \rho_2 = -\epsilon s \partial \nu / \partial s$
main references

• Magli G (1993) *Gen Rel Grav* **25**, 441
• Brito I, Carot J, Vaz EGLR (2009) submitted
Take the operator $\bar{K}^a_b = \bar{g}^{ac} \bar{h}_{cb} - \bar{u}^a \bar{u}_b$, used to measure the state of strain of the material. Then consider its eigenvalues and invariants, as follows.
Special case

So far it was not clear whether one specific spacetime metric $\bar{g}$ can be associated with different conformally related material metrics. However if one assumes that $g_{ab} = \bar{g}_{ab}$, these metrics being associated, respectively, with a flat and a non flat material metric which are conformally related, then the following relationship for constitutive equations must hold:

$$\bar{v} = \frac{1}{f^3} v.$$
Why?

This is a consequence of the following relationship between energy-momentum tensors

\[ \bar{T}_b^a = f^3_{\nu} \bar{T}_b^a \]

together with the assumption on equality of metric tensor which leads to equal energy-momentum tensors.

In this special case one also has

\[ \bar{\rho} = \rho. \]