Classical Spinning Particle Interactions in Black Hole Space-Time Backgrounds

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References:

– *Perturbation Method for Classical Spinning Particle Motion: I. Kerr Space-Time*, D. Singh, Phys. Rev. D **78** (2008) 104028; arXiv:0808.3005 [gr-qc]

– *Perturbation Method for Classical Spinning Particle Motion: II. Vaidya Space-Time*, D. Singh, Phys. Rev. D **78** (2008) 104029; arXiv:0808.3006 [gr-qc]

– *An Analytic Perturbation Approach for Classical Spinning Particle Dynamics*, D. Singh, Gen. Rel. and Grav. **40** (2008) 1179-1192; arXiv:0706.0928 [gr-qc]

Contents

1 Introduction

1.1 Classical Spinning Particles in Curved Space-Time

- An important research area in general relativity involves the dynamics of macroscopic objects with **spin angular momentum** propagating in a curved space-time background.
- Many relevant examples of classical spinning objects in a strong gravitational field in astrophysics.
  - Neutron stars in orbit around supermassive black holes,
  - Spinning particles in contact with gravitational waves, etc.
- Virtually all astrophysical sources have some angular momentum (orbital and spin), so a detailed analysis of how such objects behave in curved space-time is essential to better understand.

1.2 Mathisson-Papapetrou-Dixon (MPD) Equations

- Widely accepted that the **leading order** interaction of classical spinning objects in curved space-time is represented by the **Mathisson-Papapetrou-Dixon (MPD)** equations of motion:

$$\frac{DP^\mu}{\mathrm{d}\tau} = -\frac{1}{2} R^\mu{}_{\nu\alpha\beta} \, u^\nu \, S^{\alpha\beta} \, ,$$
$$\frac{DS^{\alpha\beta}}{\mathrm{d}\tau} = P^\alpha \, u^\beta - P^\beta \, u^\alpha \, .$$

- **Pole-Dipole Approximation:** Higher order terms due to multipole moments beyond the mass monopole and spin dipole are neglected.

1.3 Constraint Equations

- **Spin Condition:** $S^{\alpha\beta} P_\beta = 0$.
- **Parametrization Constraint:** Freedom to define parameter τ in terms of $P \cdot u$.
- **Centre-of-Mass Velocity:**

$$u^\mu = -\frac{P \cdot u}{m^2} \left[P^\mu + \frac{1}{2} \frac{S^{\mu\nu} R_{\nu\gamma\alpha\beta} P^\gamma S^{\alpha\beta}}{m^2 + \frac{1}{4} R_{\alpha\beta\rho\sigma} S^{\alpha\beta} S^{\rho\sigma}} \right].$$

- **Mass Magnitude:** $m^2 = -P_\mu P^\mu$.
- **Spin Magnitude:** $s^2 = \frac{1}{2} S_{\mu\nu} S^{\mu\nu}$.

  Both m and s are **constants of the motion** for MPD equations.

2 Perturbation Approach to MPD Equations

2.1 Formalism

- Consider a definition of P^μ and $S^{\mu\nu}$ in terms of a **perturbation order parameter** ε, such that:

$$
\begin{aligned}
P^\mu(\varepsilon) &= \sum_{j=0}^{\infty} \varepsilon^j \, P^\mu_{(j)} \, , \\
S^{\mu\nu}(\varepsilon) &= \varepsilon \sum_{j=0}^{\infty} \varepsilon^j \, S^{\mu\nu}_{(j)} = \sum_{j=1}^{\infty} \varepsilon^j \, S^{\mu\nu}_{(j-1)} \, , \\
u^\mu(\varepsilon) &\equiv \sum_{j=0}^{\infty} \varepsilon^j \, u^\mu_{(j)} \, .
\end{aligned}
$$

- For the **zeroth-order** terms:

$$\frac{DP^{\mu}_{(0)}}{\mathrm{d}\tau} = 0\,, \qquad \frac{DS^{\mu\nu}_{(0)}}{\mathrm{d}\tau} = 0\,.$$

- MPD Equations:

$$\begin{aligned} \frac{DP^{\mu}(\varepsilon)}{\mathrm{d}\tau} &= -\frac{1}{2}\, R^{\mu}{}_{\nu\alpha\beta}\, u^{\nu}(\varepsilon)\, S^{\alpha\beta}(\varepsilon)\,, \\ \frac{DS^{\alpha\beta}(\varepsilon)}{\mathrm{d}\tau} &= 2\,\varepsilon P^{[\alpha}(\varepsilon)\, u^{\beta]}(\varepsilon)\,. \end{aligned}$$

- For terms of order j:

$$\begin{aligned} \frac{DP^{\mu}_{(j)}}{\mathrm{d}\tau} &= -\frac{1}{2}\, R^{\mu}{}_{\nu\alpha\beta} \sum_{k=0}^{j-1} u^{\nu}_{(j-1-k)}\, S^{\alpha\beta}_{(k)}\,, \\ \frac{DS^{\alpha\beta}_{(j)}}{\mathrm{d}\tau} &= 2 \sum_{k=0}^{j} P^{[\alpha}_{(j-k)}\, u^{\beta]}_{(k)}\,. \end{aligned}$$

Total Mass and Spin Magnitudes as the Sum of "Radiative Corrections" to m_0 and s_0:

- Perturbed mass magnitude:

$$m^2(\varepsilon) = m_0^2 \left(1 + \sum_{j=1}^{\infty} \varepsilon^j \, \bar{m}_j^2 \right) ,$$

$$\bar{m}_j^2 = -\frac{1}{m_0^2} \sum_{k=1}^{j} P_\mu^{(j-k)} \, P^\mu_{(k)} .$$

- Perturbed spin magnitude:

$$s^2(\varepsilon) = \varepsilon^2 \, s_0^2 \left(1 + \sum_{j=1}^{\infty} \varepsilon^j \, \bar{s}_j^2 \right) ,$$

$$\bar{s}_j^2 = \frac{1}{s_0^2} \sum_{k=1}^{j} S_{\mu\nu}^{(j-k)} \, S^{\mu\nu}_{(k)} .$$

2.2 Kinematic and Dynamical Quantities

- Four-Velocity: $P \cdot u \equiv -m(\varepsilon)$

$$u^\mu(\varepsilon) = \frac{1}{m(\varepsilon)} \left[P^\mu(\varepsilon) + \frac{1}{2} \frac{S^{\mu\nu}(\varepsilon)\, R_{\nu\gamma\alpha\beta}\, P^\gamma(\varepsilon)\, S^{\alpha\beta}(\varepsilon)}{m^2(\varepsilon)\, \Delta(\varepsilon)} \right],$$

$$\Delta(\varepsilon) \equiv 1 + \frac{1}{4\, m^2(\varepsilon)}\, R_{\mu\nu\alpha\beta}\, S^{\mu\nu}(\varepsilon)\, S^{\alpha\beta}(\varepsilon)\,.$$

$$u_\mu(\varepsilon)\, u^\mu(\varepsilon) = -1 + \frac{1}{4\, m^6(\varepsilon)\, \Delta^2(\varepsilon)}\, \tilde{R}_\mu(\varepsilon)\, \tilde{R}^\mu(\varepsilon)$$

$$= -1 + O(\varepsilon^4)\,,$$

$$\tilde{R}^\mu(\varepsilon) \equiv S^{\mu\nu}(\varepsilon) R_{\nu\gamma\alpha\beta}\, P^\gamma(\varepsilon)\, S^{\alpha\beta}(\varepsilon)\,.$$

$$u^{\mu}(\varepsilon) \;=\; \sum_{j=0}^{\infty} \varepsilon^{j}\, u^{\mu}_{(j)} \;=\; \frac{P^{\mu}_{(0)}}{m_0} + \varepsilon \left[\frac{1}{m_0} \left(P^{\mu}_{(1)} - \frac{1}{2}\, \bar{m}_1^{\,2}\, P^{\mu}_{(0)} \right) \right]$$

$$+ \varepsilon^2 \left\{ \frac{1}{m_0} \left[P^{\mu}_{(2)} - \frac{1}{2}\, \bar{m}_1^2\, P^{\mu}_{(1)} - \frac{1}{2} \left(\bar{m}_2^2 - \frac{3}{4}\, \bar{m}_1^4 \right) P^{\mu}_{(0)} \right] \right.$$

$$\left. + \; \frac{1}{2m_0^3}\, S^{\mu\nu}_{(0)}\, R_{\nu\gamma\alpha\beta}\, P^{\gamma}_{(0)}\, S^{\alpha\beta}_{(0)} \right\}$$

$$+ \varepsilon^3 \left\{ \frac{1}{m_0} \left[P^{\mu}_{(3)} - \frac{1}{2}\, \bar{m}_1^2\, P^{\mu}_{(2)} - \frac{1}{2} \left(\bar{m}_2^2 - \frac{3}{4}\, \bar{m}_1^4 \right) P^{\mu}_{(1)} \right. \right.$$

$$\left. - \; \frac{1}{2} \left(\bar{m}_3^2 - \frac{3}{2}\, \bar{m}_1^2\, \bar{m}_2^2 + \frac{5}{8}\, \bar{m}_1^6 \right) P^{\mu}_{(0)} \right]$$

$$\left. + \; \frac{1}{2m_0^3}\, R_{\nu\gamma\alpha\beta} \left[\sum_{n=0}^{1} S^{\mu\nu}_{(1-n)} \sum_{k=0}^{n} P^{\gamma}_{(n-k)}\, S^{\alpha\beta}_{(k)} - \frac{3}{2}\, \bar{m}_1^2\, S^{\mu\nu}_{(0)}\, P^{\gamma}_{(0)}\, S^{\alpha\beta}_{(0)} \right] \right\} + O(\varepsilon^4) \, .$$

- Spin-to-Mass Ratio (Møller radius):

$$\rho(\varepsilon) = \frac{s(\varepsilon)}{m(\varepsilon)} = \frac{s_0}{m_0}\left\{\varepsilon + \varepsilon^2\left[\frac{1}{2}\left(\bar{s}_1^2 - \bar{m}_1^2\right)\right]\right.$$

$$\left. + \varepsilon^3\left[\frac{1}{2}\left(\bar{s}_2^2 - \bar{m}_2^2\right) - \frac{1}{4}\,\bar{s}_1^2\,\bar{m}_1^2 - \frac{1}{8}\left(\bar{s}_1^4 - 3\,\bar{m}_1^4\right)\right] + O(\varepsilon^4)\right\}.$$

$$\bar{s}_1^2 = \frac{2}{s_0^2}\,S^{(1)}_{\mu\nu}\,S^{\mu\nu}_{(0)}\,, \qquad \bar{s}_2^2 = \frac{1}{s_0^2}\left[2\,S^{(2)}_{\mu\nu}\,S^{\mu\nu}_{(0)} + S^{(1)}_{\mu\nu}\,S^{\mu\nu}_{(1)}\right].$$

$$\bar{m}_1^2 = -\frac{2}{m_0^2}\,P^{(1)}_\mu\,P^\mu_{(0)} = 0\,,$$

$$\bar{m}_2^2 = -\frac{1}{m_0^2}\left[2\,P^{(2)}_\mu\,P^\mu_{(0)} + P^{(1)}_\mu\,P^\mu_{(1)}\right],$$

$$\bar{m}_3^2 = -\frac{2}{m_0^2}\left[P^{(3)}_\mu\,P^\mu_{(0)} + P^{(2)}_\mu\,P^\mu_{(1)}\right].$$

$$\frac{D\bar{s}_j^2}{d\tau} = \frac{D\bar{m}_j^2}{d\tau} = 0\,, \qquad S^{\mu\nu}\,P_\nu = 0\,.$$

3 Application to Black Hole Space-Time Backgrounds

Orthonormal Tetrad: $\{\lambda^\mu{}_{\hat{\alpha}}\}$, $\qquad \eta_{\hat{\alpha}\hat{\beta}} = g_{\mu\nu}\, \lambda^\mu{}_{\hat{\alpha}}\, \lambda^\nu{}_{\hat{\beta}}$

$$\begin{aligned} \frac{D\lambda^\mu{}_{\hat{\alpha}}}{\mathrm{d}\tau} &= 0\,, \\ {}^F R_{\hat{\alpha}\hat{\beta}\hat{\gamma}\hat{\delta}} &= R_{\mu\nu\rho\sigma}\, \lambda^\mu{}_{\hat{\alpha}}\, \lambda^\nu{}_{\hat{\beta}}\, \lambda^\rho{}_{\hat{\gamma}}\, \lambda^\sigma{}_{\hat{\delta}}\,. \end{aligned}$$

$$\begin{aligned} P^\mu_{(0)} &= m_0\, u^\mu_{(0)} \;=\; m_0\, \lambda^\mu{}_{\hat{0}}\,, \\ S^{\mu\nu}_{(0)} &= \lambda^\mu{}_{\hat{\imath}}\, \lambda^\nu{}_{\hat{\jmath}}\, S^{\hat{\imath}\hat{\jmath}}_{(0)}\,, \qquad \left(S^{\mu\nu}_{(0)}\, P^{(0)}_\nu \;=\; 0\right) \end{aligned}$$

- Initial spin orientation $(\hat{\theta}, \hat{\phi})$ for $S^{\mu\nu}_{(0)}$ coincides with the standard definition for the spherical co-ordinates (θ, ϕ) with respect to the z-axis of the black hole's body-frame.

3.1 Kerr Space-Time

Boyer-Lindquist co-ordinates: $X^\mu = (t, r, \theta, \phi)\,, \quad \Omega_{\rm K} \; = \; \sqrt{M/r^3}$.

$$\begin{aligned}
{\lambda^\mu}_{\hat{0}} &= \left(\frac{1 + a\,\Omega_{\rm K}}{N}, 0, 0, \frac{\Omega_{\rm K}}{N\,\sin\theta}\right), \\
{\lambda^\mu}_{\hat{1}} &= \left(-\frac{L}{r\,A}\,\sin\left(\Omega_{\rm K}\,\tau\right),\, A\,\cos\left(\Omega_{\rm K}\,\tau\right),\, 0,\, -\frac{E}{r\,A\,\sin\theta}\,\sin\left(\Omega_{\rm K}\,\tau\right)\right), \\
{\lambda^\mu}_{\hat{2}} &= \left(0, 0, \frac{1}{r}, 0\right), \\
{\lambda^\mu}_{\hat{3}} &= \left(\frac{L}{r\,A}\,\cos\left(\Omega_{\rm K}\,\tau\right),\, A\,\sin\left(\Omega_{\rm K}\,\tau\right),\, 0,\, \frac{E}{r\,A\,\sin\theta}\,\cos\left(\Omega_{\rm K}\,\tau\right)\right).
\end{aligned}$$

$$\begin{aligned}
E &= \frac{1}{N}\left(1 - \frac{2M}{r} + a\,\Omega_{\rm K}\right), \qquad L = \frac{r^2\Omega_{\rm K}}{N}\left(1 - 2a\,\Omega_{\rm K} + \frac{a^2}{r^2}\right), \\
A &= \sqrt{1 - \frac{2M}{r} + \frac{a^2}{r^2}}\,, \qquad N = \sqrt{1 - \frac{3M}{r} + 2a\,\Omega_{\rm K}}\,.
\end{aligned}$$

3.2 Vaidya Space-Time

Metric: $X^\mu = (\xi, r, \theta, \phi)\,, \quad \Omega_{\rm K} \;=\; \sqrt{M_0/r^3}.$

$$\begin{aligned} {\rm d}s^2 &= -\left(1 - \frac{2\,M(\xi)}{r}\right) {\rm d}\xi^2 + 2\,\alpha\,{\rm d}\xi\,{\rm d}r + r^2\left({\rm d}\theta^2 + \sin^2\theta\,{\rm d}\phi^2\right)\,, \\ M(\xi) &= M_0 + \Delta M(\xi)\,, \\ \xi &= t + \alpha\left[r + 2M_0\,\ln\left(\frac{r}{2M_0} - 1\right)\right]. \end{aligned}$$

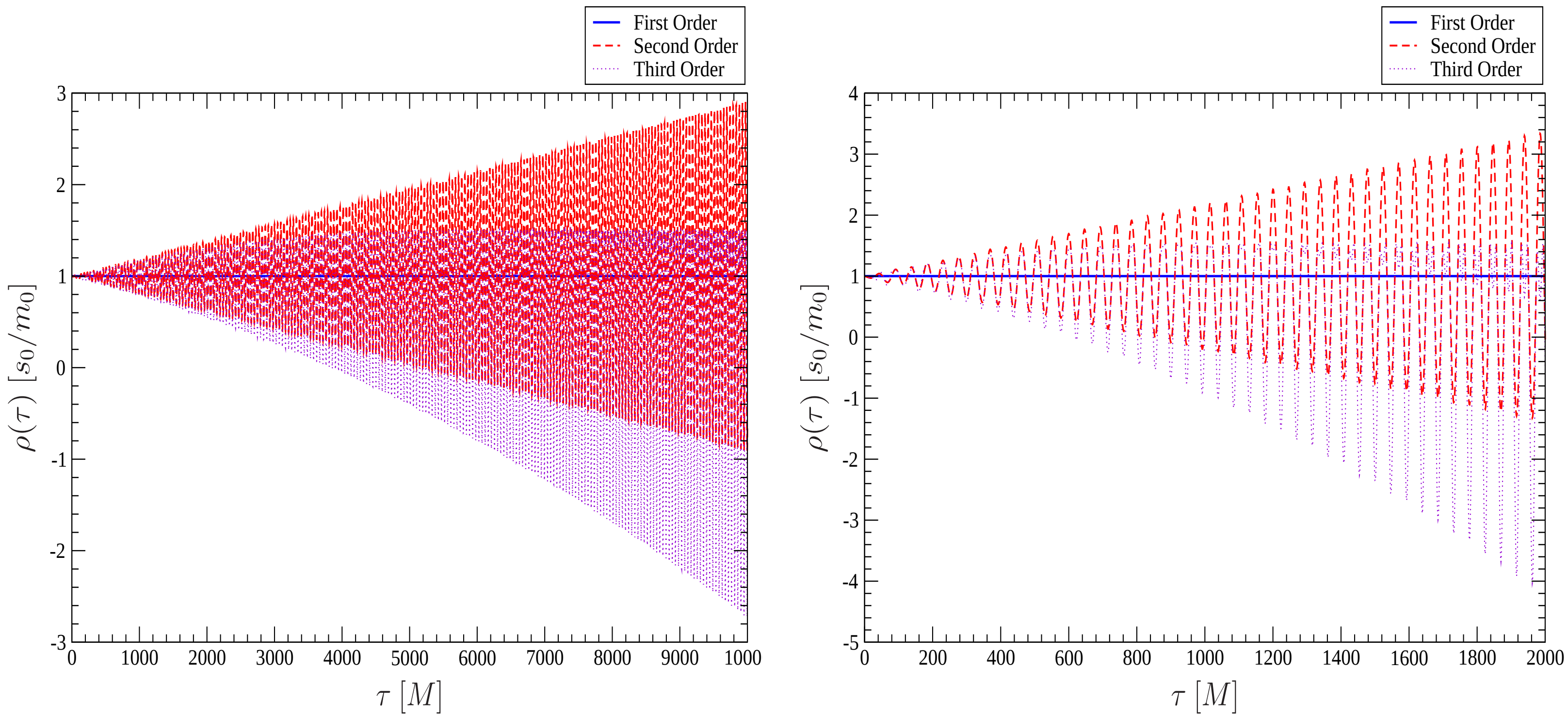
$\alpha = +1$ (infalling radiation)

$\alpha = -1$ (outgoing radiation)

Orthonormal Tetrad:

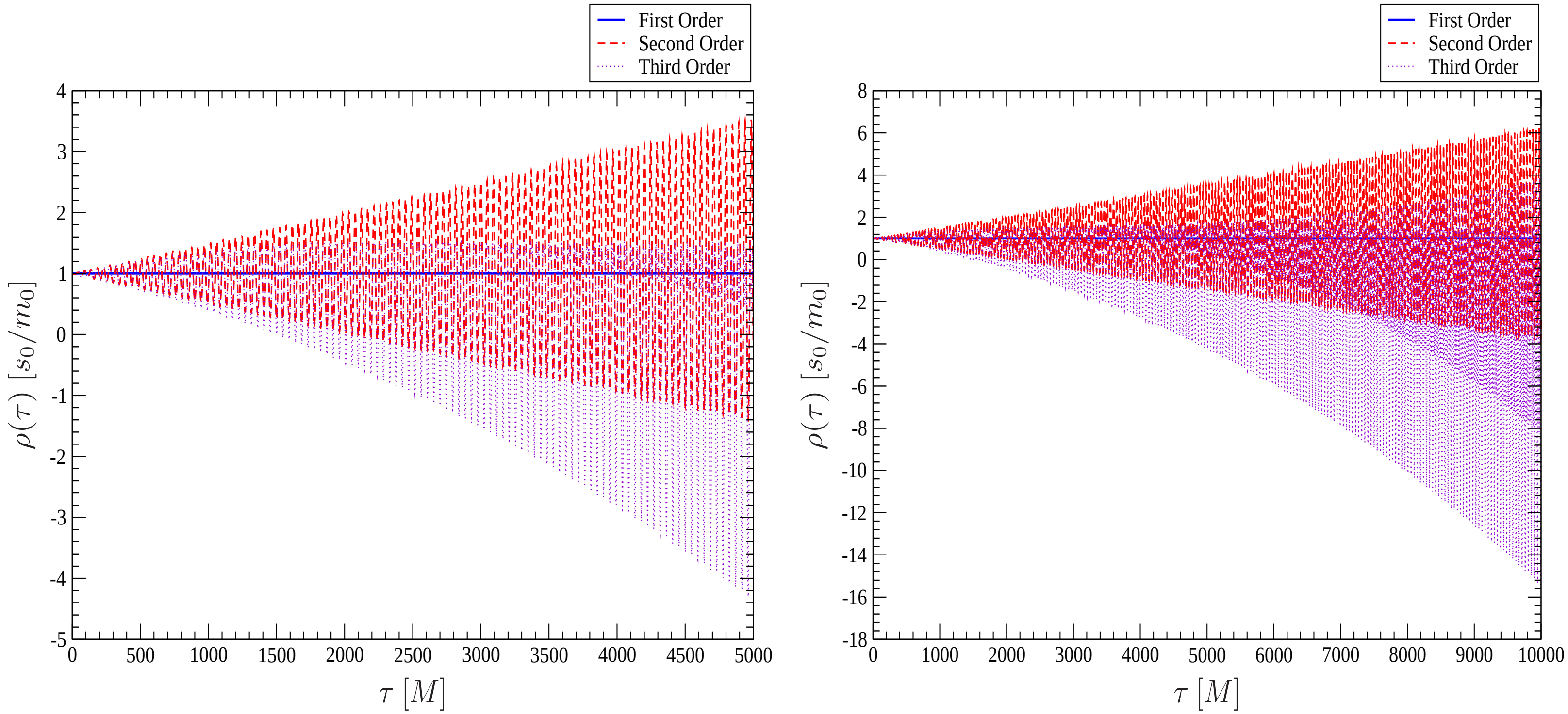
$$\lambda^\mu{}_{\hat\alpha} \;\approx\; \lambda^\mu{}_{\hat\alpha\,({\rm Sch})} + \Delta\lambda^\mu{}_{\hat\alpha}\,.$$

3.3 Orbital Stability Analysis in the Kerr and Vaidya Backgrounds



(a) $s_0/(m_0\, r) = 10^{-1}\, ,\ a = M$ (b) $s_0/(m_0\, r) = 10^{-1}\, ,\ a = -M$

Figure 1: Møller radius $\rho(\tau) = (s/m)(\tau)$ in the Kerr background for $r = 6M$ and $\hat{\theta} = \hat{\phi} = \pi/4$, in units of s_0/m_0.



(a) $s_0/(m_0\, r) = 10^{-1}\,,\ \alpha = 1$

(b) $s_0/(m_0\, r) = 10^{-1}\,,\ \alpha = -1$

Figure 2: Møller radius $\rho(\tau) = (s/m)(\tau)$ in the Vaidya background for $r = 6M_0$ and $\hat{\theta} = \hat{\phi} = \pi/4$, in units of s_0/m_0.

4 Conclusion

- A systematic perturbation approach for analytically describing classical spinning objects in curved space-time is presented, based on the MPD equations of motion, with potentially interesting consequences.
- Formalism appears to be robust and is self-consistent up to $O(\varepsilon^3)$.
- Has the possibility of contributing to many research directions.

Examples of Various Potential Research Directions:

- Gravitomagnetic analysis in the Kerr background.
- Scattering due to gravitational waves.
- Analysis of transition from stable to chaotic motion.
- Interaction between two or more spinning particles.
  - Continuum limit?
- Modifications of gravitational self-force and outgoing gravitational radiation from a point dipole due to spin-gravity coupling?