On self-gravitating elementary solutions of non-linear electrodynamics

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Overview of the topic

- Hoffman and Infeld (1935-37) found solutions to the Einstein equations coupled to non-linear electrodynamics (NED).
- 80's: Renewed interest on the topic, mainly motivated by string theory and effective field theories:
  - García-Salazar-Plebanski'84, Demianski´86, Oliveira'94, Gibbons+Rasheed'95, Rasheed'97: Born-Infeld-like models coupled to gravitation → Black hole and ``particle-like" solutions
- Different generalizations of Einstein-Born-Infeld theories :
  - Inclusion of a dilaton field: Tamaki'00, Clement'00
  - Asymptotically AdS spaces: Fernando'03'06
  - N-dimensions: Cai'04,Dey'04
  - Non-abelian fields
- Other NED solutions:
  - Regular magnetic black hole solutions: Bronnikov'00
  - ``Mixed" electric solutions with a magnetic (Burinski'02) core or a de Sitter one (Dymnikova'04)
  - Regular, electrically charged black hole solutions (Ayón-Beato'99'00)

NED in gravitation

$$F = -\frac{1}{2} F_{\mu\nu} F^{\mu\nu}$$
$$G = -\frac{1}{2} F_{\mu\nu} F^{*\mu\nu}$$

- Einstein-NED action: $S = \int d^4x \sqrt{|g|} \left(\frac{R}{16\pi G} - L(F,G) \right)$

- For static, spherically symmetric NED solutions the metric can be written as

$$ds^2 = g(r)dt^2 - g(r)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

and for purely electrostatic solutions (ESS) $(\vec{A}(r) = 0, \vec{E}(r) = -\vec{\nabla} A_0(r))$ the Einstein-NED equations can be integrated to give:

$(F = A_0'^2, G = 0)$

$$g(r) = 1 - \frac{2M}{r} + \frac{1}{r}\int_r^\infty \tilde{r}^2 T_0^0(\tilde{r}) d\tilde{r}$$

where

$$T_0^0 = T_1^1 = 2L_F A_0'^2 - L(F) \;;\; T_2^2 = T_3^3 = -L(F)$$

- First-integral: $r^2 L_F A_0'^2 = q$ Same form as in the flat-space case

$\nabla_\mu (L_F F^{\mu\nu}) = 0$

- Behaviour of the metric depends on the sign of the quantity: $2M - \varepsilon$
  where

$$\varepsilon(q) = \int_0^\infty r^2 T_0^0(r)dr = q^{3/2}\varepsilon(q=1)$$

  is related to the total electromagnetic energy in *Minkowski* space.

- We consider NED models such that
  I. a set of minimal physical requirements (``admissibility" conditions) is satisfied and
  II. support (flat-space) finite-energy ESS fields.

(details on Annals. Phys. 324 (2009) 827)

I. Admissibility: Vanishing vacumm energy + positive definite energy for *any* field configuration $\rho_S = 2\left(L_F \vec{E}^2 + L_G \vec{E}\cdot\vec{B}\right) - L(F,G)$

Necessary conditions:

$$L(0,0) = 0; L(F,0) < 0 \forall (F<0, G=0); L_F > 0 \forall (F,G)$$

$$\rho_S \geq \left(\sqrt{F^2+G^2}+F\right)L_F + GL_G - L(F,G) \geq 0$$

⇒ WEC automatically holds for the admissible models

II. Finite-energy conditions: Convergence governed by the behaviour of the energy functional near the limits of the integral

- At infinity (B-cases):

$$A_0'(r) \approx \frac{\alpha}{r^p}, p > 1$$ B-1 (1<p<2), B-2 (p=2), B-3 (p>2)

- At the origen three possible behaviours:

A-1: $A_0'(r) \approx C r^{\sigma}, -1 < \sigma < 0$

A-2: $A_0'(r) \approx C - \theta r^{\sigma}, \sigma > 0$

A-3: $A_0'(r) \approx C r^{\sigma}, \sigma > 0 \rightarrow$ **Forbidden by the admissibility constraints**

Important: $r^2 T_0^0(r)$ is a monotonically decreasing function of r for finite-energy ESS solutions of admissible models.

Analysis of the black hole solutions

A. Metric

$$\underline{r \to \infty}: \; g(r) \to 1 - \frac{2M}{r} + \frac{4}{(p-1)(p+2)} \frac{q\alpha}{r^p}, \; p > 1$$

$$\underline{r \to 0}; A-1: \;\; g(r) \approx 1 - \frac{2M - \varepsilon}{r} + \frac{4Cq}{(\sigma-2)(\sigma+1)} r^{\sigma}, -1 < \sigma < 0$$

Subcases: $2M - \varepsilon \geq 0 : g(r) \to -\infty :$ 1 horizon

$2M - \varepsilon < 0 : g(r) \to +\infty :$ 2, 1 (extreme) or zero horizons

$$\underline{r \to 0}; A-2: \;\; g(r) \approx 1 - \frac{2M - \varepsilon}{r} - 2qC - \frac{4q\theta}{(\sigma-2)(\sigma+1)} r^{\sigma} + \frac{\Delta r^2}{3}, \sigma > 0$$

Subcases: $2M - \varepsilon > 0 : g(r) \to -\infty :$ 1 horizon

$2M - \varepsilon < 0 : g(r) \to +\infty :$ 2, 1 (extreme) or zero horizons

$2M - \varepsilon = 0 : g(r) \to 1 - 2qC :$

1 (2qC>1) or zero (2qC<1) horizons

B. Mass-Horizon radius

- Horizons given by $g(r_h) = 0 \Rightarrow r_h = 2M - \int_{r_h}^{\infty} r^2 T_0^0(r) dr$

  Let us analyze the black hole mass as a function of the horizon radius

$$\boxed{M(r_h) = \frac{1}{2}\left(\varepsilon + r_h - \int_0^{r_h} r^2 T_0^0(r) dr \right)}$$

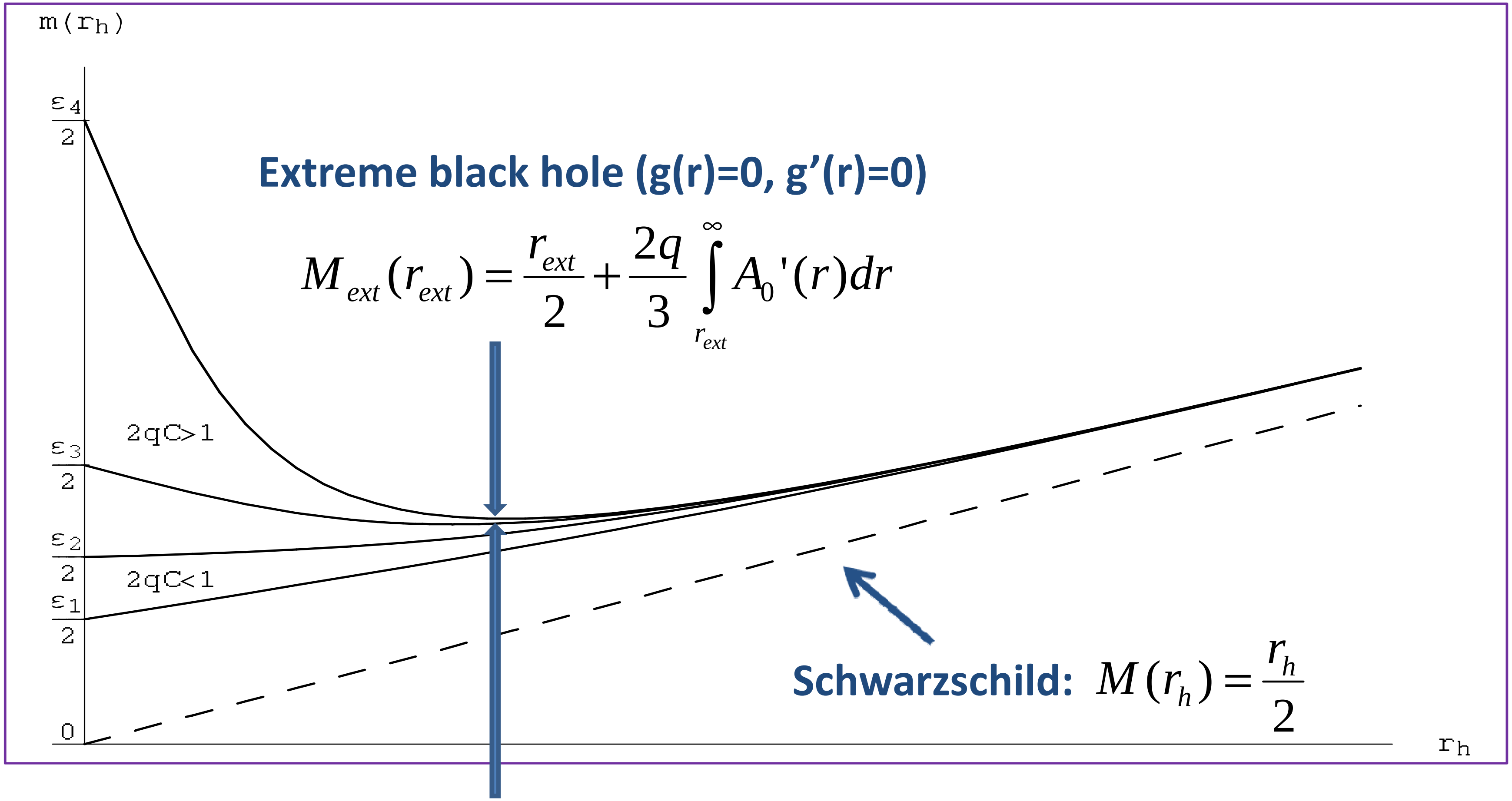
- Asymptotically the behaviour becomes

$$\underline{r_h \to \infty}: \quad M(r_h) \approx \frac{r_h}{2} + \frac{2}{(p-1)(p+2)} \frac{q\alpha}{r_h^{\ p-1}}, p > 1$$

$$\underline{r_h \to 0}: \quad \begin{array}{l} A-1: M(r_h) \approx \dfrac{\varepsilon}{2} + \dfrac{r_h}{2} + \dfrac{2Cq}{(\sigma+1)(\sigma-2)} r_h^{\ \sigma+1}, -1 < \sigma < 0 \\ A-2: M(r_h) \approx \dfrac{\varepsilon}{2} + \dfrac{(1-2qC)}{2} r_h - \dfrac{2qC}{(\sigma+1)(\sigma-2)} r_h^{\sigma+1} + \dfrac{\Delta r_h^3}{6}, \sigma > 0 \end{array}$$

Analysis of the black hole solutions:

B. Mass-Horizon radius



Extreme black hole (g(r)=0, g'(r)=0)

$$M_{ext}(r_{ext}) = \frac{r_{ext}}{2} + \frac{2q}{3}\int_{r_{ext}}^{\infty} A_0'(r)dr$$

Schwarzschild: $M(r_h) = \frac{r_h}{2}$

$$r_{ext}^2 T_0^0(r_{ext}) = 1$$

C. <u>Thermodynamics</u>

- Temperature of the black holes can be determined through the definition of the surface gravity for our solutions:

$$k = \frac{1}{2}\frac{dg(r)}{dr}\bigg|_{r=r_h}$$

- Temperature: $$\boxed{T = \frac{k}{2\pi} = \frac{1}{4\pi r_h}\left(1 - r_h^2 T_0^{\ 0}(r_h)\right)}$$

- Asymptotically:

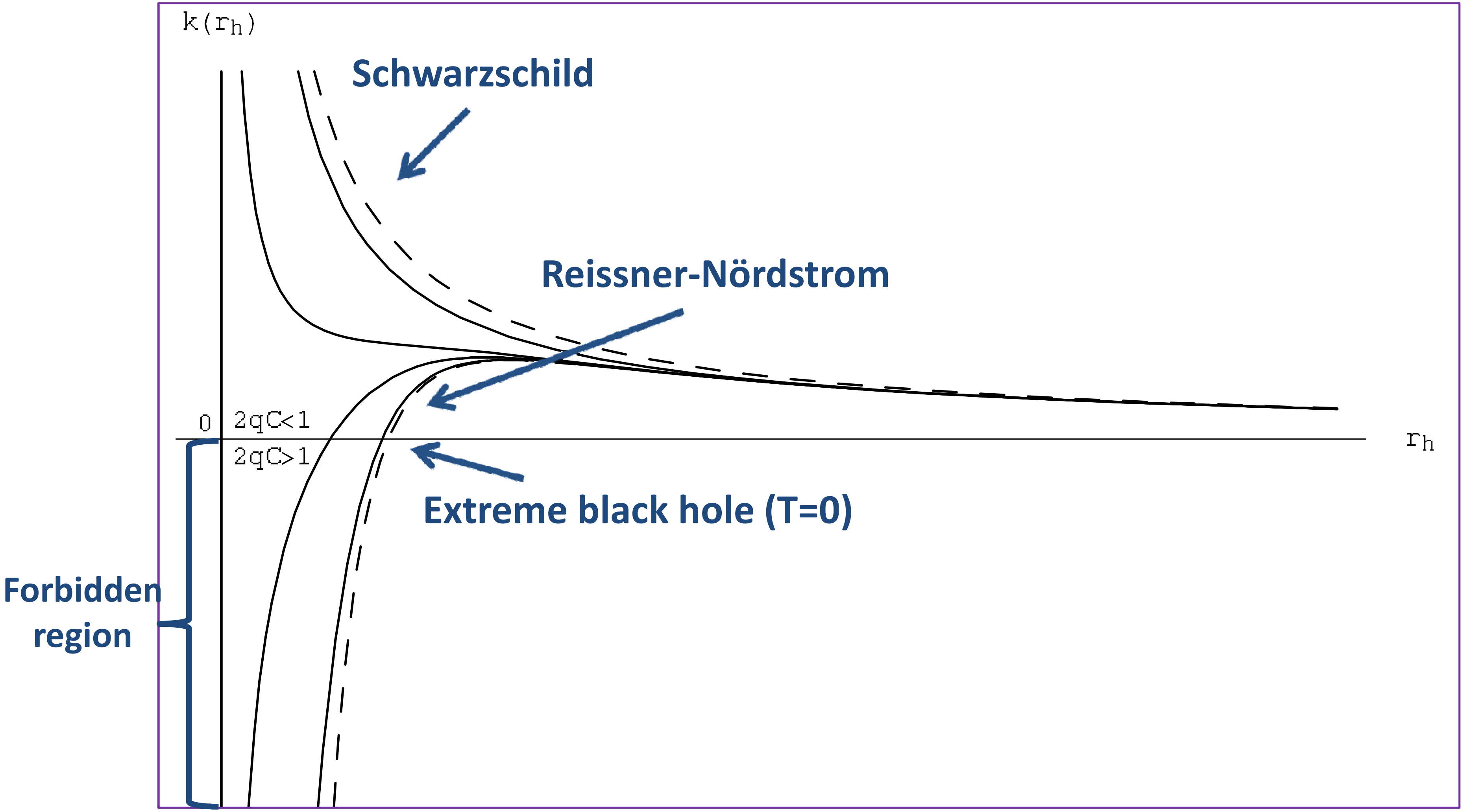
$$\underline{r \to \infty}: \quad T \approx \frac{1}{4\pi}\left(\frac{1}{r_h} - \frac{4\alpha q}{p+2}\frac{1}{r_h^{p+1}}\right), p > 1$$

$$A-1: T \approx \frac{1}{4\pi}\left(\frac{1}{r_h} + \frac{4qC}{\sigma - 2} r_h^{\sigma-1}\right), -1 < \sigma < 0$$

$\underline{r \to 0}$:

$$A-2: T \approx \frac{1}{4\pi}\left(\frac{(1-2qC)}{r_h} - \frac{4q\theta}{\sigma - 2} r_h^{\sigma-1} + \Delta r_h\right)$$

Analysis of the black hole solutions:
C. Thermodynamics



Conclusions and open problems

- Properties of black holes in Einstein-NED strongly depend on the relation between the ADM mass and the flat-space soliton energy
- Several classes of field behaviours near origin: A-1 (the field diverges, the energy not), A-2 (it takes a finite value), A-3 (discarded)

- Issues to be investigated in more detail:
  - Stability: Linear perturbation analysis in flat-space leads to a supplementary condition for the finite-energy ESS solutions:

    $$\boxed{L_F - 2FL_{GG} > 0 \forall (F > 0, G = 0)}$$ **0809.0684 [hep-th]**

    → Extension of this criterium to the black hole stability?

    (See also gr-qc/0208090)
  - Electric + magnetic solutions: Regularity?
  - Extension of the formalism to more general cases: inclusion of dilaton field, n-dimensions, (A)dS spaces…