

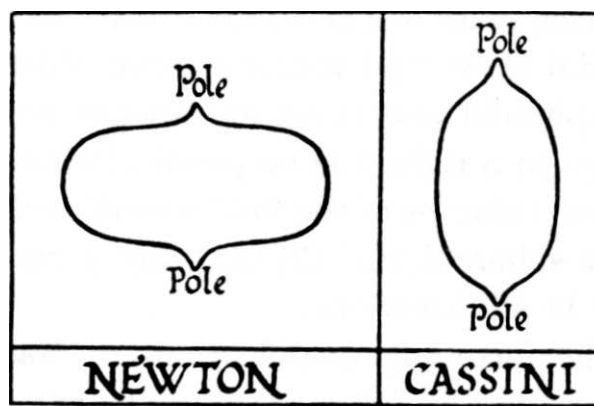
Relativistic figures of equilibrium: from Maclaurin spheroids to Kerr black holes

Reinhard Meinel, University of Jena

1. Equilibrium figures of rotating fluids within Newton's theory of gravity
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1. Equilibrium figures of rotating fluids within Newton's theory of gravity

Newton, Maclaurin, Jacobi, Liouville, Dirichlet, Dedekind, Riemann, Poincaré, F. W. Dyson, Kowalewskaja, H. Cartan, Lichtenstein, Chandrasekhar



Newton (1686):

$$\frac{\text{equatorial radius} - \text{polar radius}}{\text{mean radius}} \approx \frac{5\Omega^2 R^3}{4GM} \approx 0.0043$$

Actual value: 0.0034

First empirical evidence: *Maupertuis & Clairaut (1738)*

L. Lichtenstein (1933), *Gleichgewichtsfiguren rotierender Flüssigkeiten*, Springer, Berlin

S. Chandrasekhar (1969), *Ellipsoidal figures of equilibrium*, Yale University Press, New Haven

Interior:

$$\Delta U = 4\pi G \mu, \quad \nabla p = -\mu \nabla V, \quad V = U - \frac{1}{2} \Omega^2 \rho^2$$

Exterior:

$$\Delta U = 0$$

Surface:

$$p = 0 \quad \Rightarrow \quad V = V_0 = \text{constant}$$

For a given relation $p = p(\mu)$ or $\mu = \mu(p)$ the solution depends only on two parameters:

$$V_0, \Omega \quad (\text{or, for example: } M, J)$$

Homogeneous bodies: $\mu = \text{constant}$

2. Maclaurin spheroids and Dyson rings

Maclaurin spheroids

Maclaurin (1742):

$$\frac{\Omega^2}{\pi G \mu} = \frac{2}{\epsilon^3} \sqrt{1 - \epsilon^2} (3 - 2\epsilon^2) \arcsin \epsilon - \frac{6}{\epsilon^2} (1 - \epsilon^2)$$

with $\epsilon = \sqrt{1 - r_p^2/r_e^2}$

$\epsilon \ll 1$: → Newton's formula for the oblateness

$\epsilon \rightarrow 1$: → Maclaurin disc ("disc of dust")

Characteristic eccentricities ϵ :

0.81267 Jacobi bifurcation point

0.95289 Onset of non-axisymmetric dynamical instability

0.99856 Onset of axisymmetric dynamical instability

Bifurcation points of axisymmetric sequences:

$$\epsilon_1 = 0.98523, \epsilon_2 = 0.99375, \epsilon_3 = 0.99657,$$

$$\epsilon_4 = 0.99784, \epsilon_5 = 0.99851, \epsilon_6 = 0.99891, \dots$$

Dyson rings

Kowalewskaja (1885), Poincaré (1885):

$$\frac{\Omega^2}{\pi G \mu} \rightarrow \frac{1}{4} (1 - \rho_i/\rho_o)^2 \ln \frac{16}{1 - \rho_i/\rho_o} \quad \text{as} \quad \rho_i/\rho_o \rightarrow 1$$

Frank Dyson (1892, 1893):

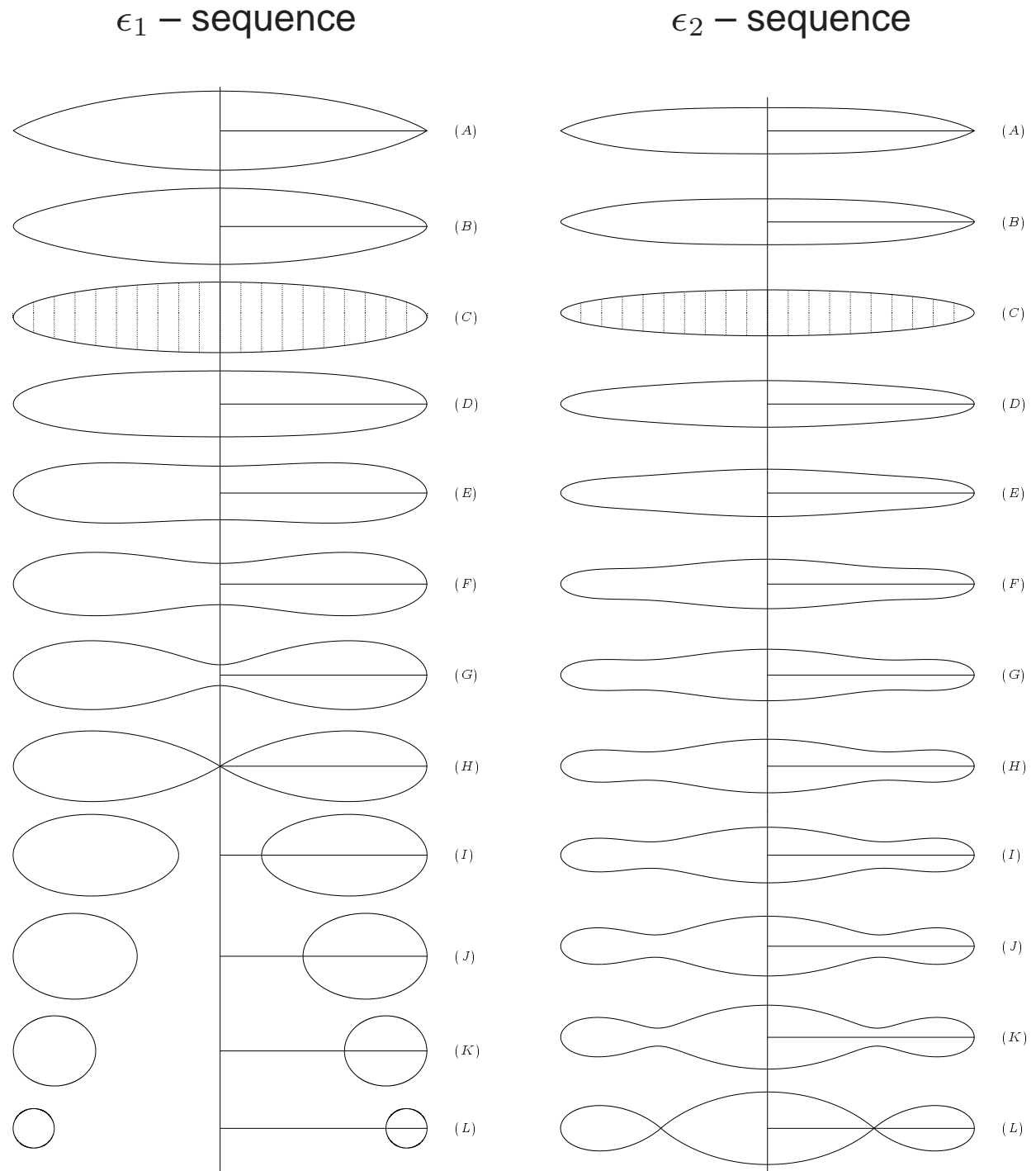
Expansion to higher orders

Bardeen (1971), Eriguchi & Sugimoto (1981):

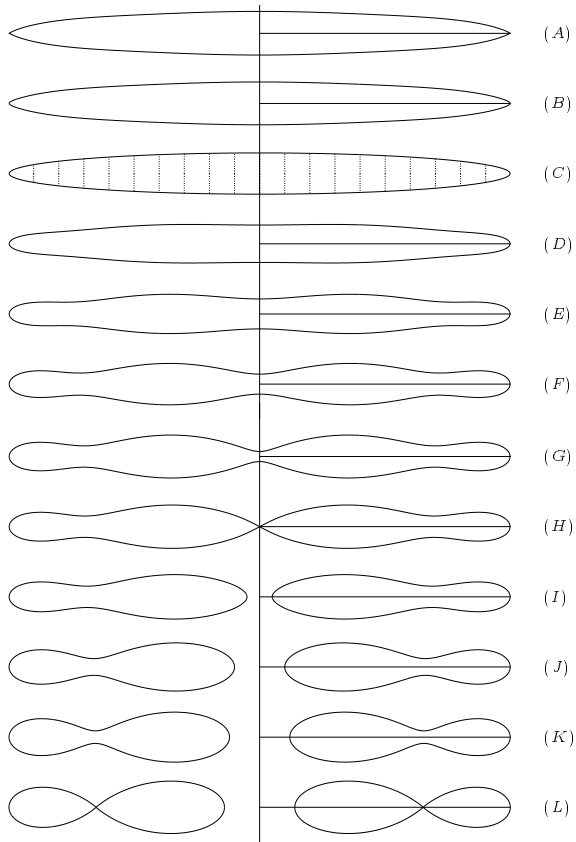
Connection of the Dyson rings with the Maclaurin spheroids

3. Further axisymmetric sequences

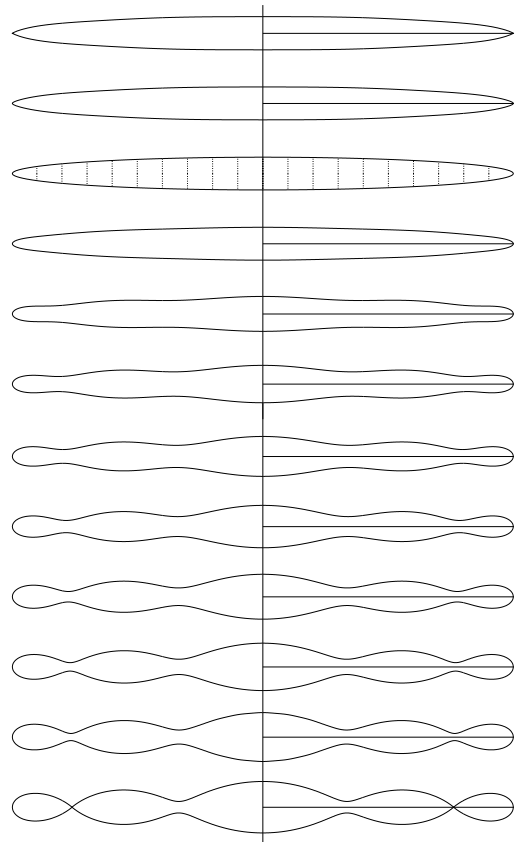
[Ansorg, Kleinwächter & M., MNRAS 339 (2003) 515]



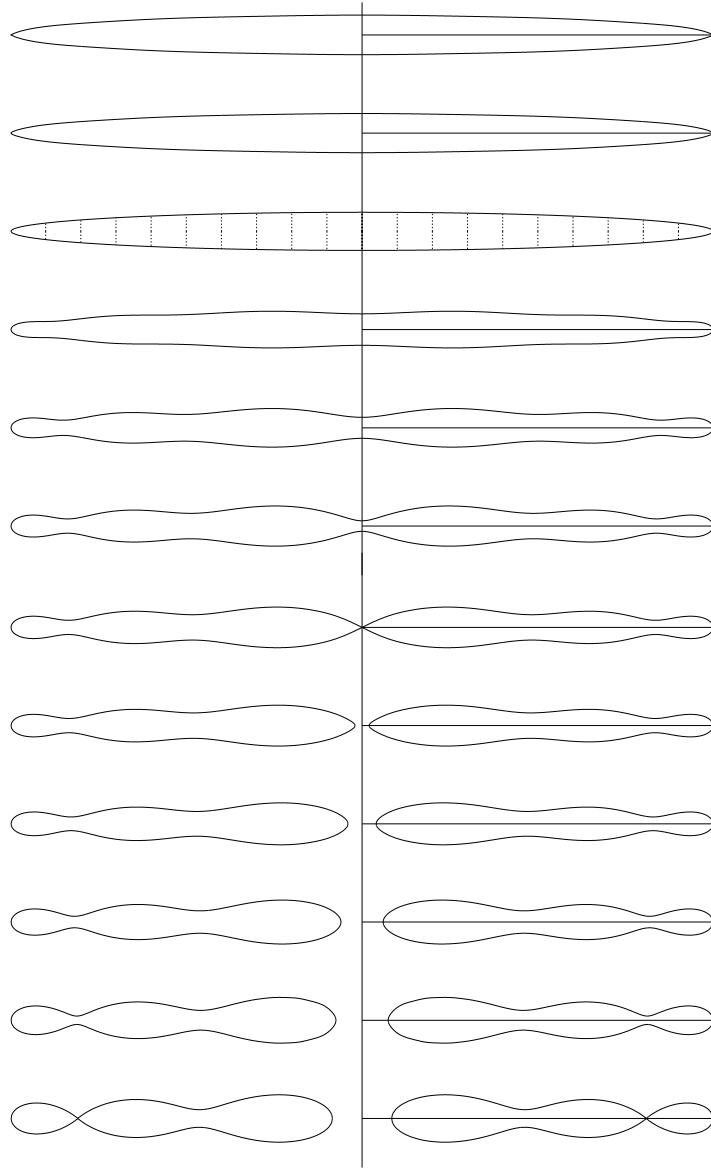
ϵ_3 – sequence



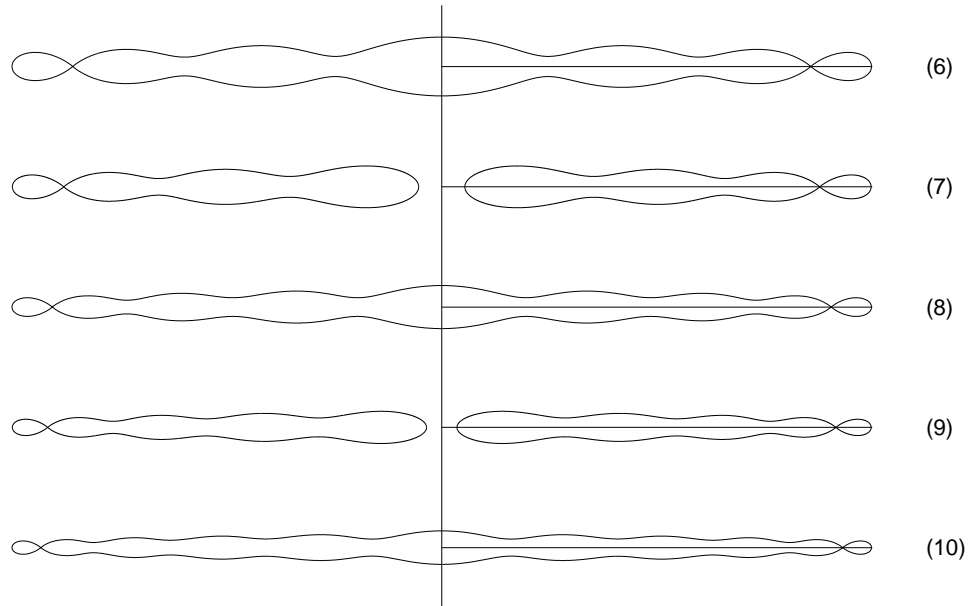
ϵ_4 – sequence



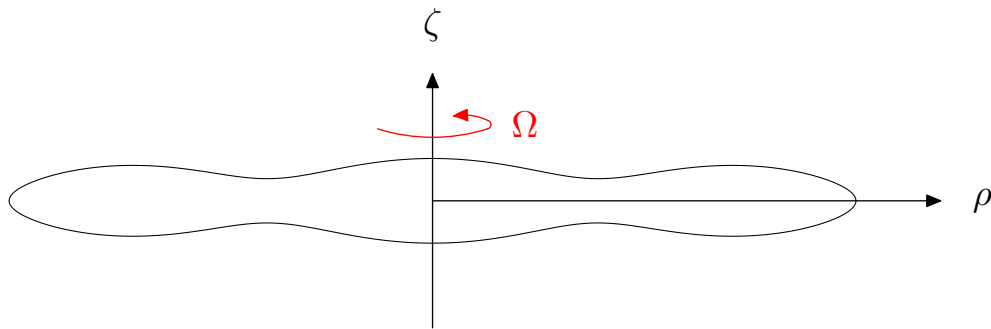
ϵ_5 – sequence



$\epsilon_6 - \epsilon_{10}$ – sequences (fragmentation limit)



4. Equilibrium fluid configurations within Einstein's theory



Two Killing vectors: $\xi = \frac{\partial}{\partial t}$, $\eta = \frac{\partial}{\partial \varphi}$

$$ds^2 = e^{2\alpha}(d\rho^2 + d\zeta^2) + W^2 e^{-2\nu}(d\varphi - \omega dt)^2 - e^{2\nu} dt^2$$

$$R_{ik} - \frac{1}{2}R g_{ik} = 8\pi T_{ik} \quad (G = c = 1)$$

Interior:

$$T_{ik} = (\mu + p) u_i u_k + p g_{ik}, \quad \mu = \mu(p)$$

$$u^i = e^{-V} (\xi^i + \Omega \eta^i), \quad \Omega = \text{constant}$$

$$\Rightarrow \nabla p = -(\mu + p) \nabla V$$

Exterior:

$$T_{ik} = 0$$

Surface:

$$p = 0 \quad \Rightarrow \quad V = V_0 = \text{constant}$$

Two parameters: V_0, Ω (or, for example: M, J)

Relative redshift of zero angular momentum photons ($\eta_i p^i = 0$) emitted from the surface and received at infinity:

$$Z_0 = e^{-V_0} - 1$$

Homogeneous bodies: $\mu = \text{constant}$

Analytically known limiting cases:

- Schwarzschild spheres ($\Omega = 0$)
- Maclaurin spheroids ($Z_0 \ll 1$)
- Relativistic discs of dust ($p_{\text{max}}/\mu \rightarrow 0$)

Analytical results on post-Newtonian Maclaurin spheroids:

*Chandrasekhar (1967), Bardeen (1971);
D. Petroff, PhD thesis, Jena (2003)*

- Singularities at $\epsilon_1, \epsilon_2, \dots$

First numerical results for $\mu = \text{constant}$:

Butterworth & Ipser (1975, 1976), Butterworth (1979)

Review on numerical methods [arbitrary $\mu(p)$] in:

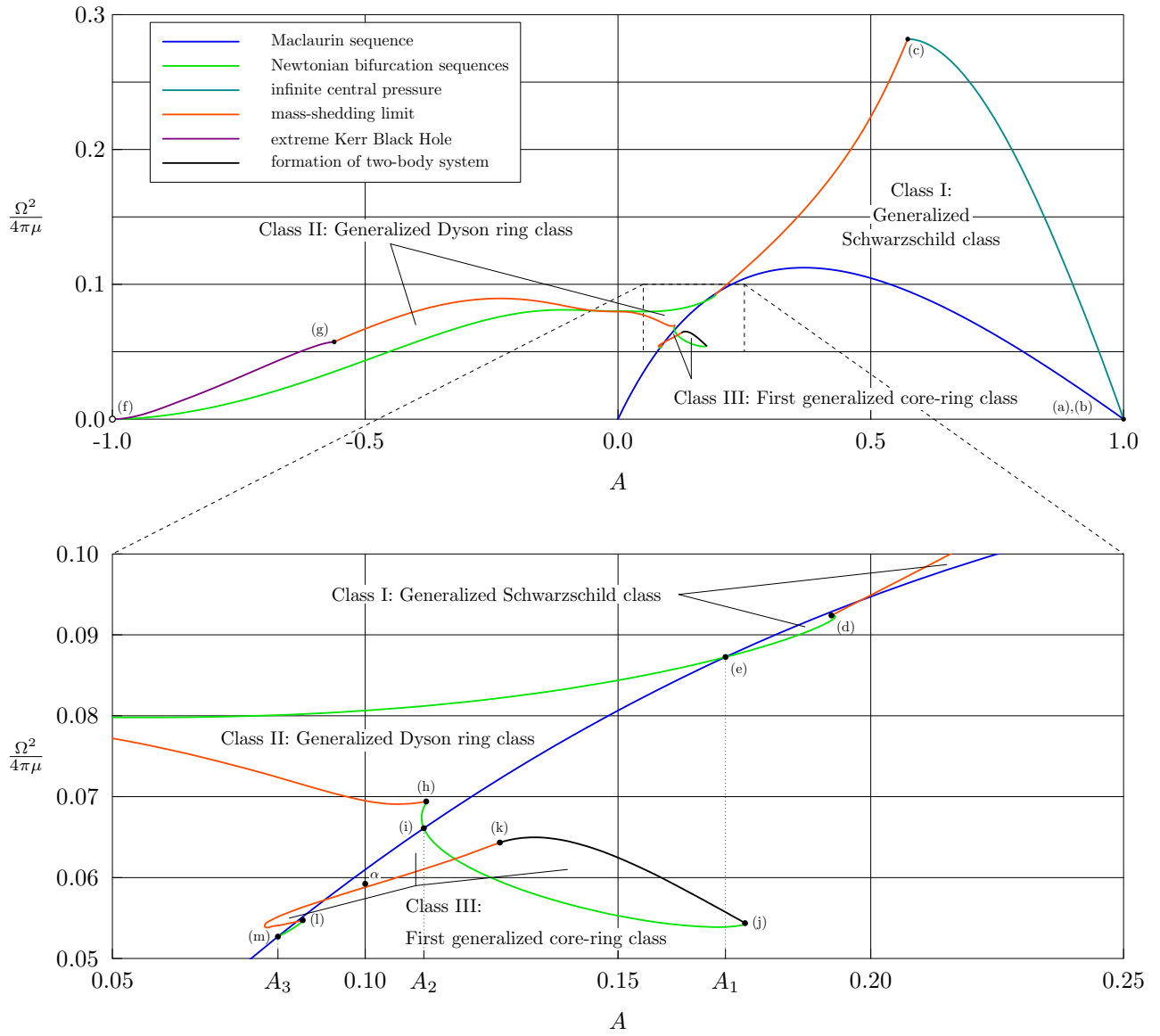
N. Stergioulas, Rotating stars in relativity, Living Reviews in Relativity 2003-3

- By means of a multi-domain spectral method one can reach machine accuracy!

[Ansorg, Kleinwächter & M., Astronomy & Astrophysics 381 (2002) L49, 405 (2003) 711]

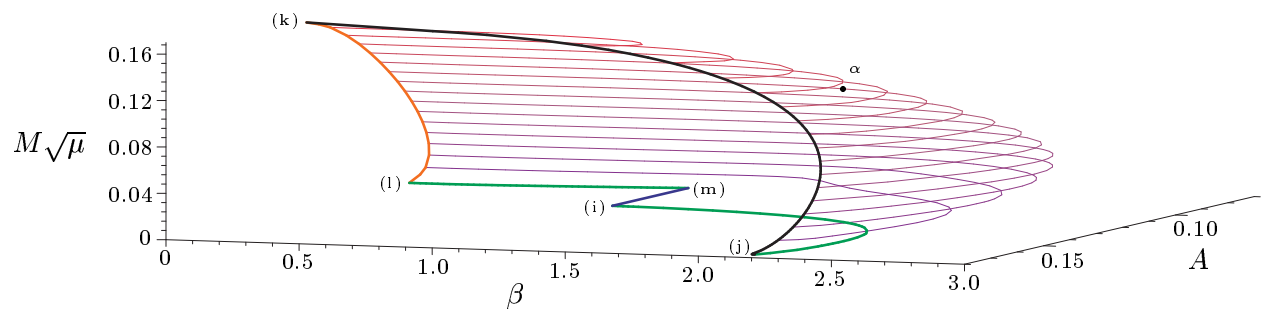
Overview of the solutions for $\mu = \text{constant}$:

Ansorg, Fischer, Kleinwächter, M., Petroff & Schöbel, MNRAS 355 (2004) 682



$$A := \frac{r_1}{r_2} \begin{cases} r_1 = \text{polar radius} & \text{(spheroidal case)} \\ r_2 = \text{equatorial radius} \\ -r_1 = \text{inner radius} & \text{(toroidal case)} \\ r_2 = \text{outer radius} \end{cases}$$

Maclaurin spheroids: $\epsilon = \sqrt{1 - A^2}$

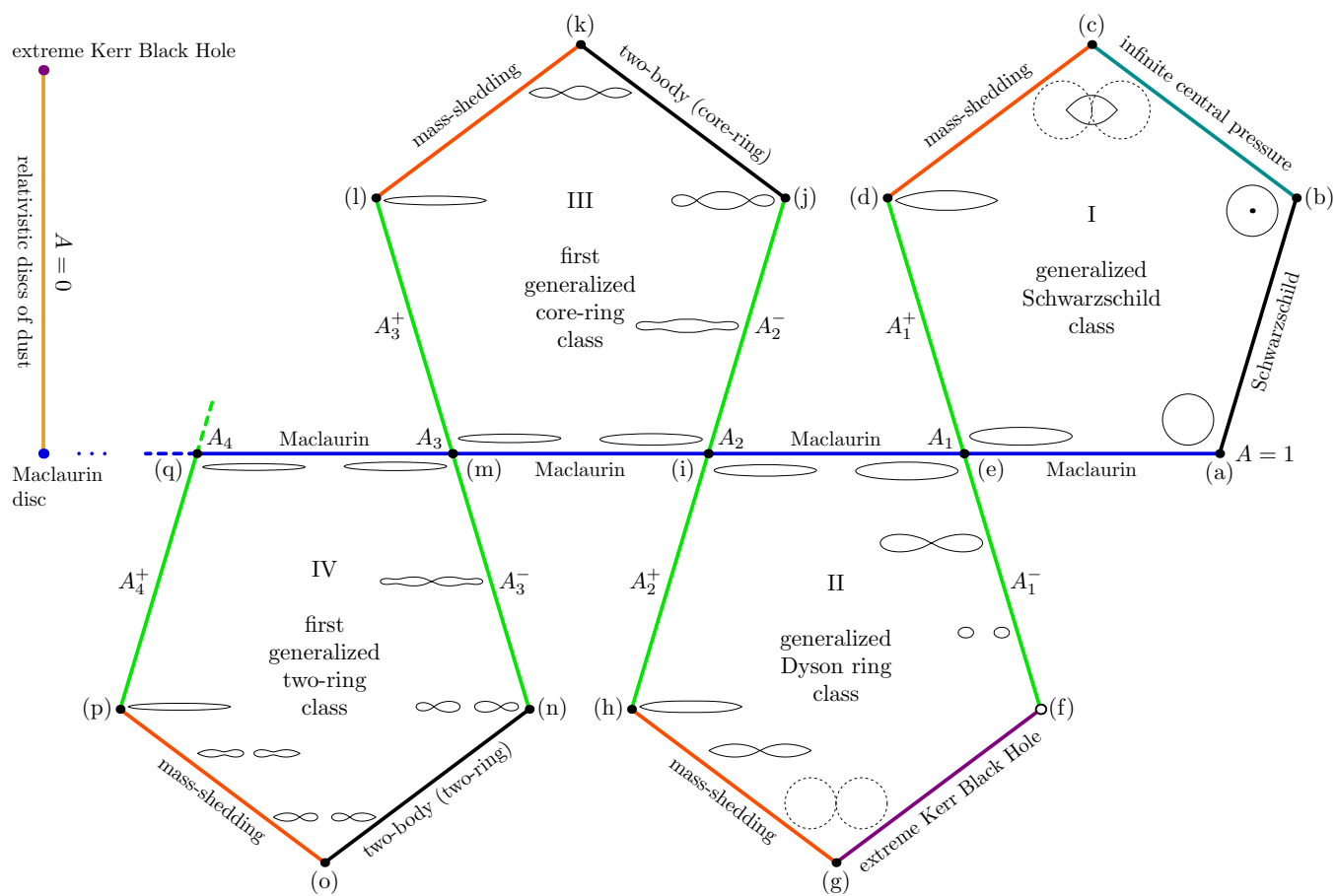


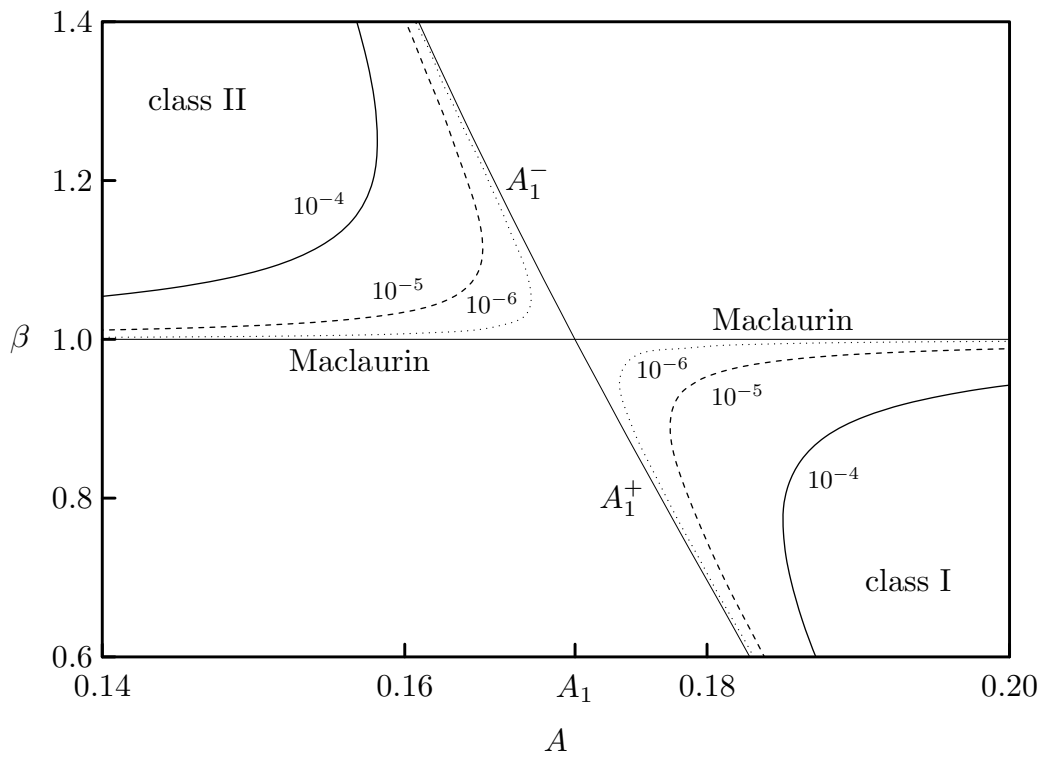
Mass-shedding parameter β :

$$\beta := -\frac{r_2}{r_1^2} \lim_{\rho \rightarrow r_2} \zeta_s \frac{d\zeta_s}{d\rho},$$

[boundary of the configuration: $\zeta = \zeta_s(\rho)$]

$\beta = 0$ in the mass-shedding limit,
 $\beta = 1$ for Maclaurin spheroids

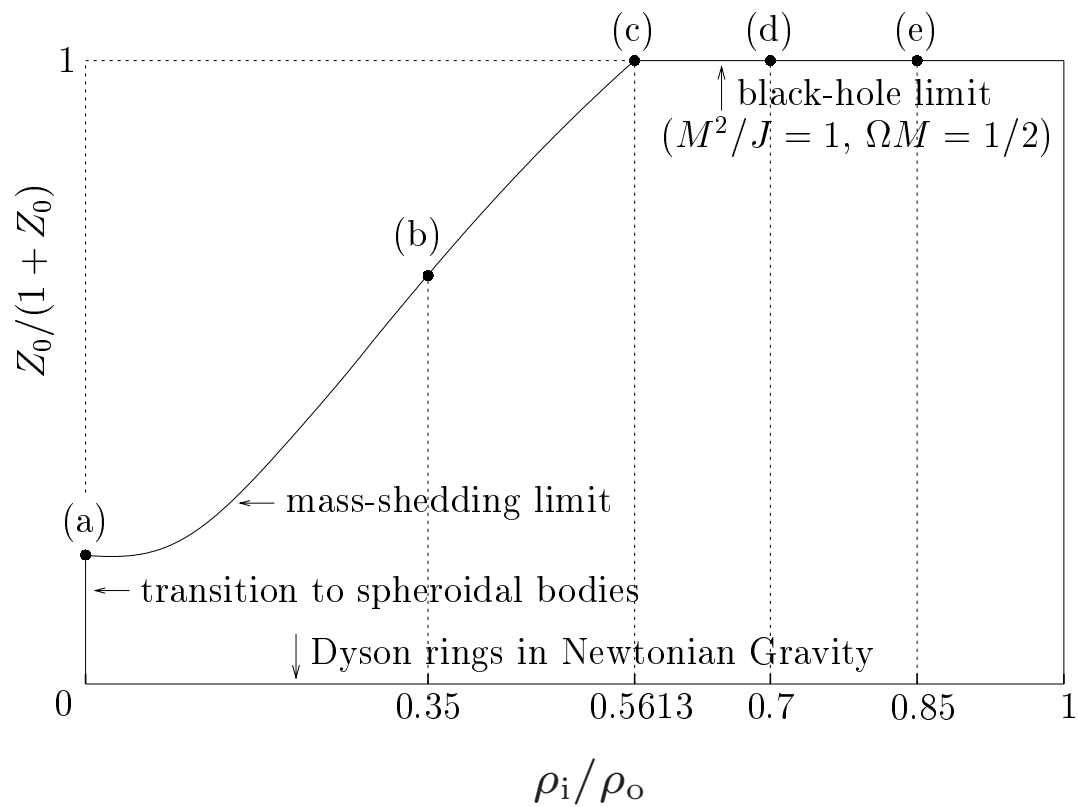




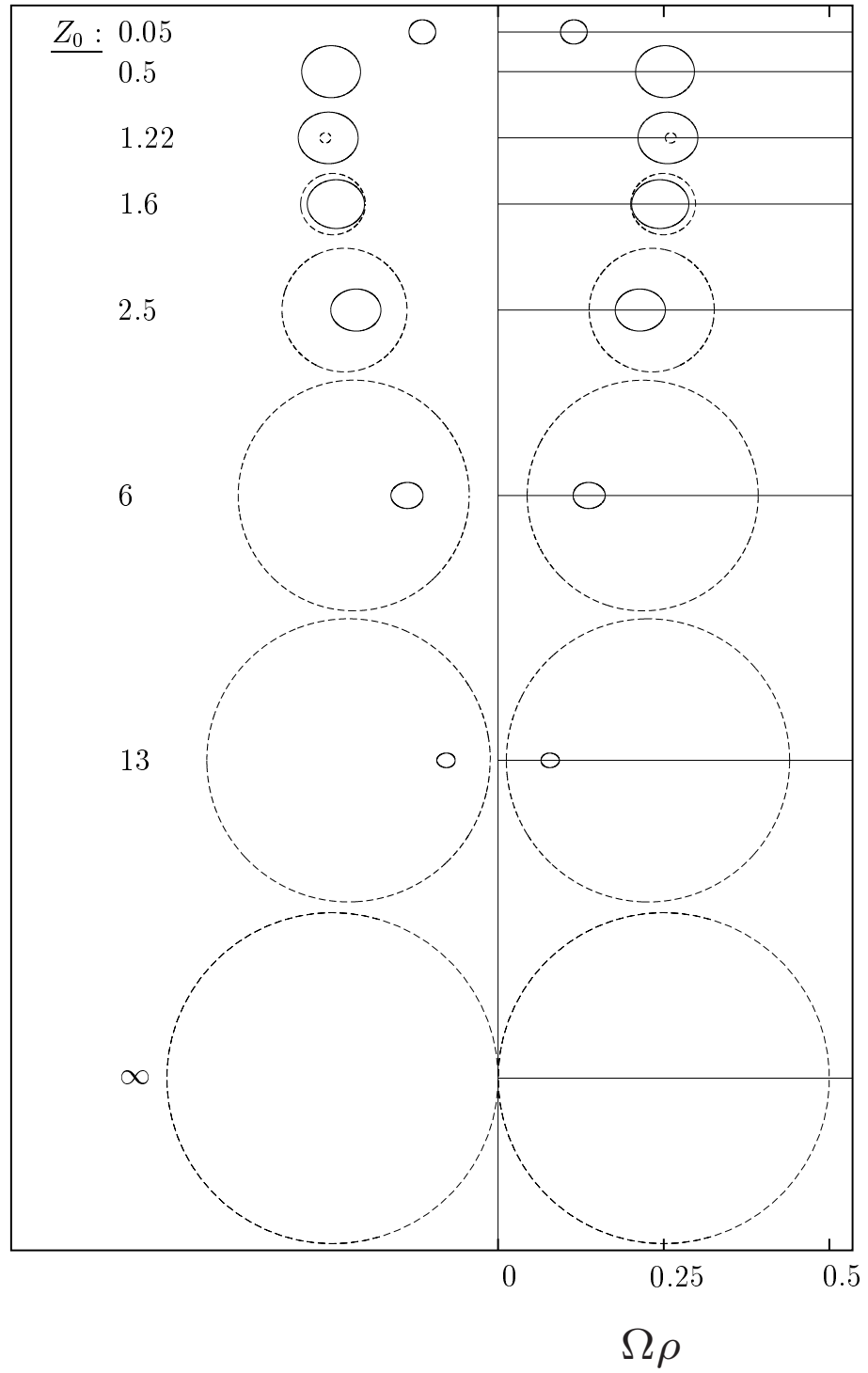
Sequences of constant $M\sqrt{\mu}$ in the vicinity of the bifurcation point A_1

5. Relativistic Dyson rings and their black hole limit

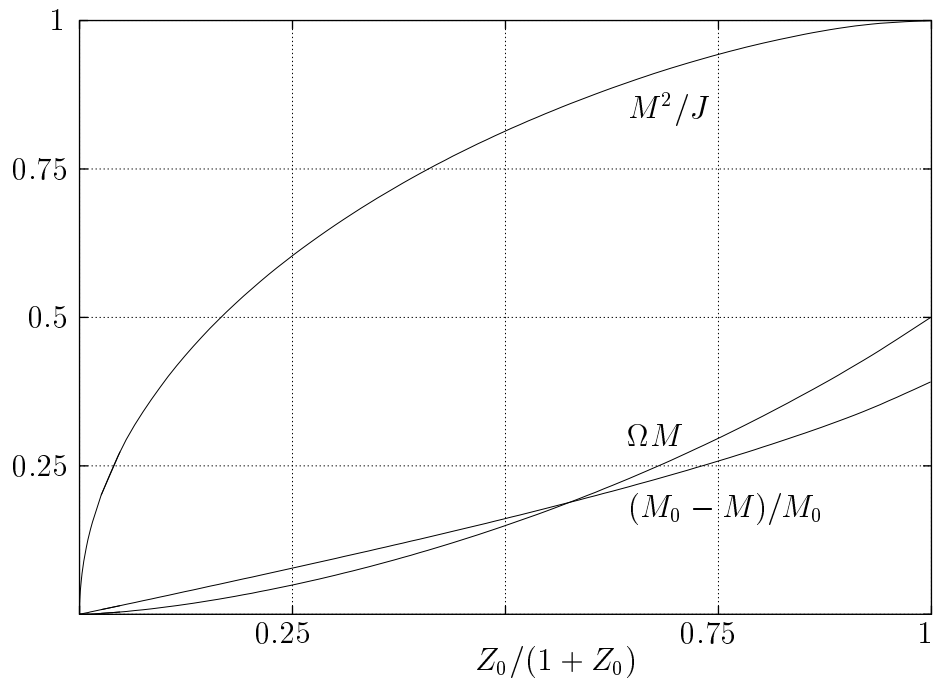
[Ansorg, Kleinwächter & M., *ApJ* 582 (2003) L87]



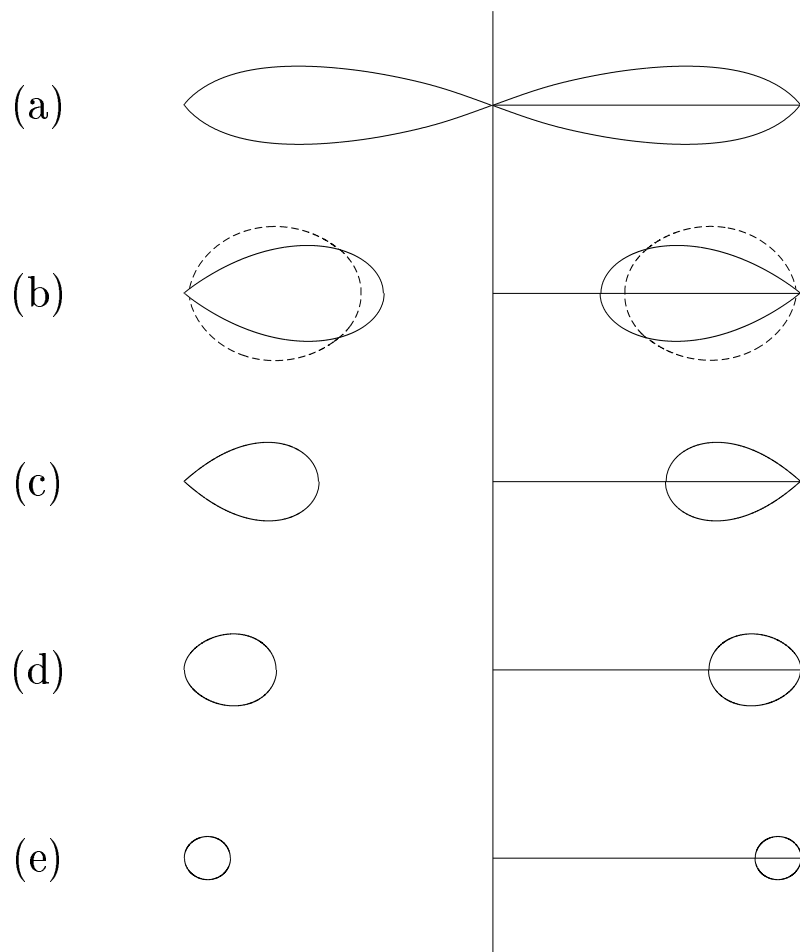
$\Omega\zeta$



$$\rho_i/\rho_o = 0.7$$

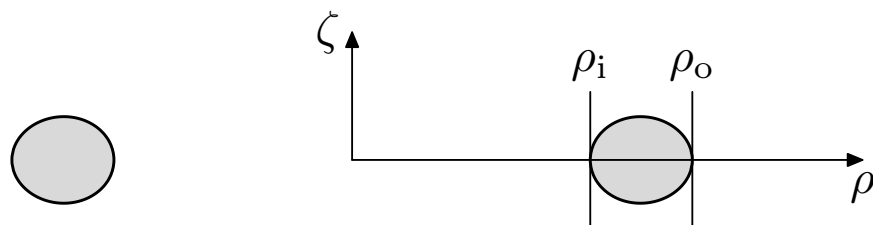
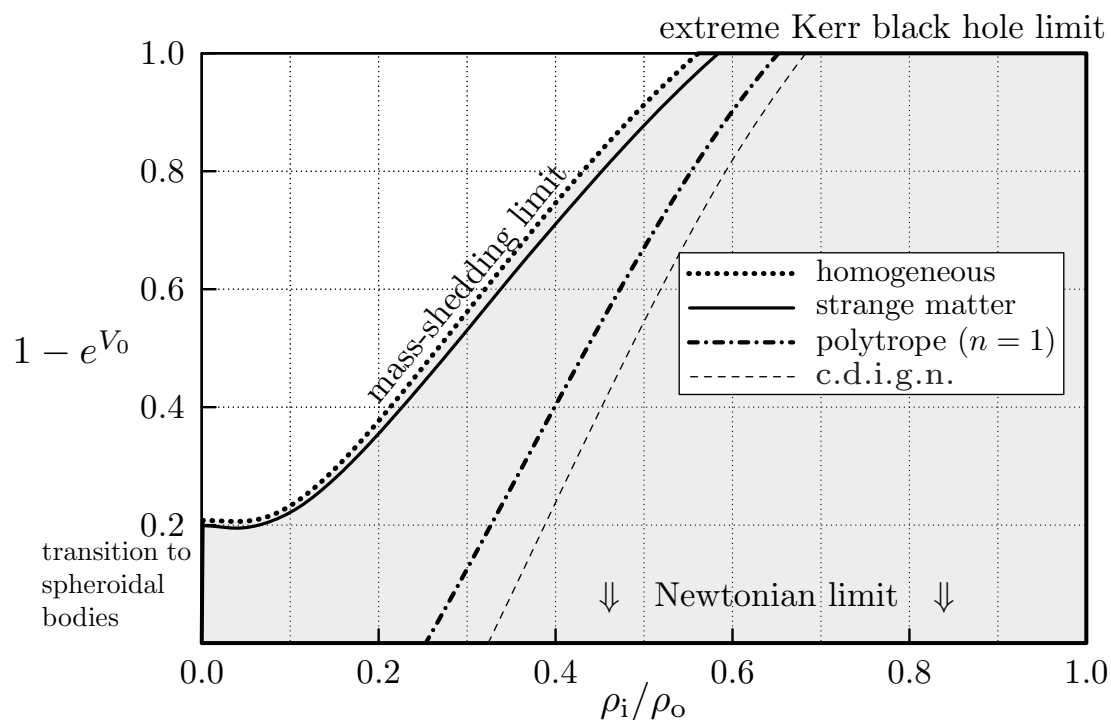


$$\rho_i/\rho_o = 0.7$$

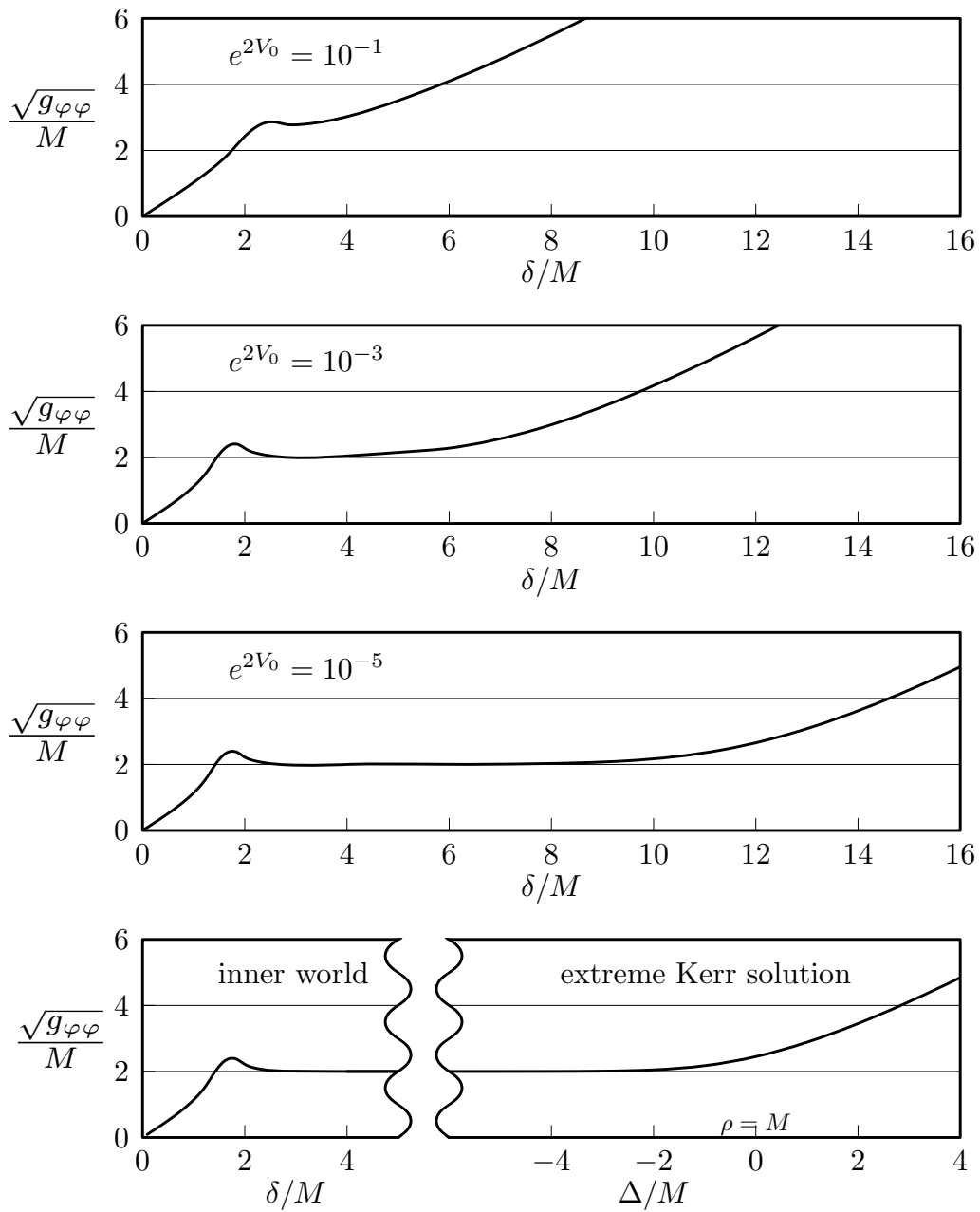


6. Rings with other equations of state

[Fischer, Horatschek & Ansorg, *MNRAS* 364 (2005) 943;
 Labranche, Petroff & Ansorg, *Gen. Rel. Grav.* 39 (2007) 129]



Meridional cross-section of a “strange quark matter” ring
 ($\mu = 3p + 4B$) with $\rho_i/\rho_o = 0.7$ and $e^{2V_0} = 0.1$

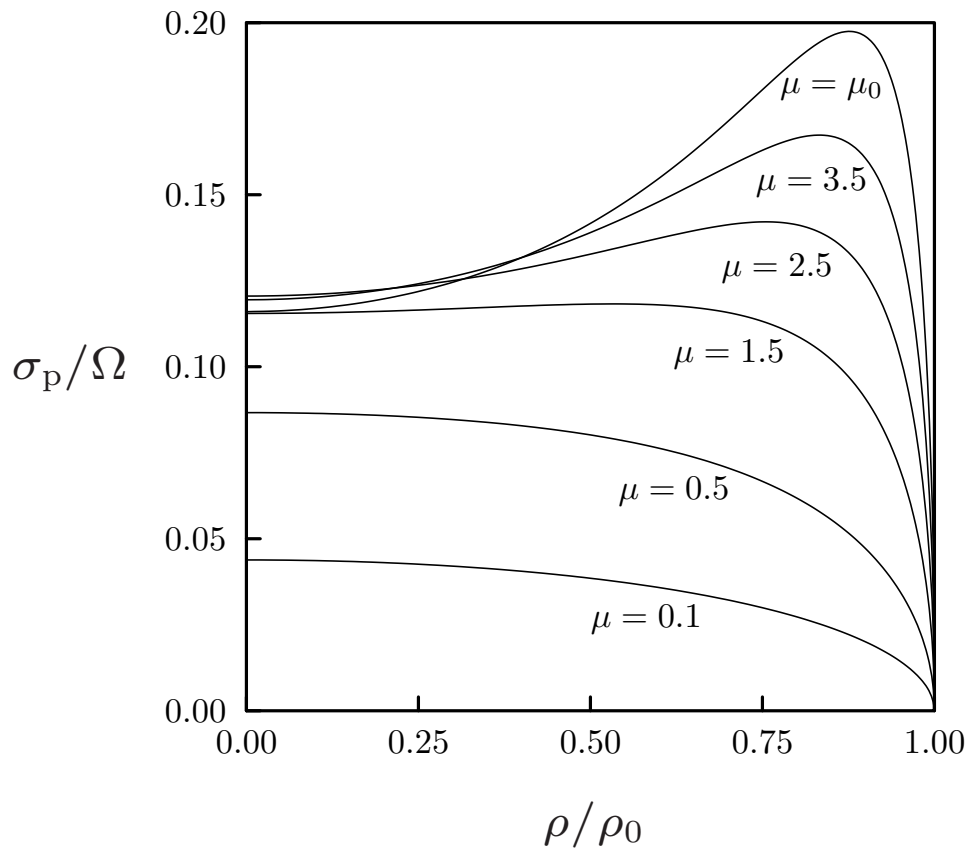


Sequence of “strange quark matter” rings ($\rho_i/\rho_o = 0.7$) with transition to a black hole, $\zeta = 0$,
 δ : proper radial distance to the center,
 Δ : proper radial distance to the outer rim of the ergosphere

7. The disc limit

[Bardeen & Wagoner (1971);

Neugebauer & M., *Phys. Rev. Lett.* 75 (1995) 3046]



Here: $\mu = 2\Omega^2 \rho_0^2 e^{-2V_0}$, relativity parameter

$\mu \ll 1$: Maclaurin disc

$\mu \rightarrow \mu_0 = 4.62966184 \dots$: transition to black hole
(with $J = M^2$)

[R. M., *Ann. Phys. (Leipzig)* 11 (2002) 509]

8. A necessary and sufficient condition for a (quasi) black hole limit of rotating fluid bodies in equilibrium

[R. M., *Ann. Phys. (Leipzig)* 13 (2004) 600;
Class. Quant. Grav. 23 (2006) 1359]

A quasi-stationary (parametric) transition of a fluid configuration to a black hole occurs if and only if the gravitational mass becomes equal to twice the product of angular velocity and angular momentum:

$$M = 2\Omega J$$

Consequences:

- (1) No quasi-stationary transition without rotation
- (2) $J = M^2$ (maximally rotating Kerr black hole)

Relativistic Figures of Equilibrium

R. Meinel, M. Ansorg, A. Kleinwächter, G. Neugebauer & D. Petroff
Cambridge University Press 2008