The spherically symmetric body in relativistic elasticity

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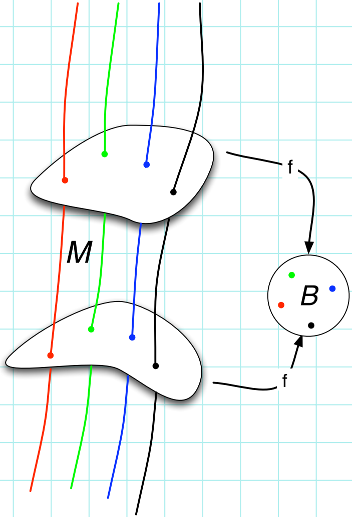
Motivation

- Conceptual problem:
  How to generalise the classical theory of elasticity to a relativistic framework?
- Application:
  Elastic properties of bar detectors
- Neutron stars have an elastic crust.
  Is there an influence on gravitational waves(?)

Geometry and Kinematics

Basic variable



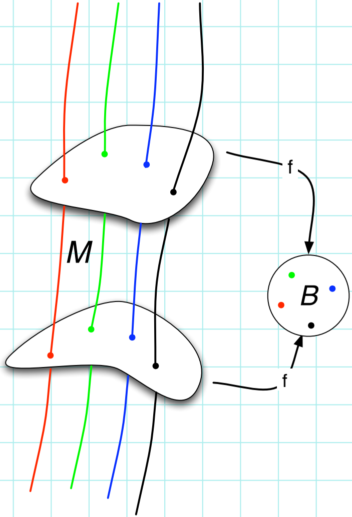
- body in space-time M
- body consists of ideal particles
- particle traces world-line
- fundamental variable

$$f : M \to B, \quad x^a \mapsto X^A = f^A(x^a)$$

- collapses world-lines into points

Geometry and Kinematics

Basic variable



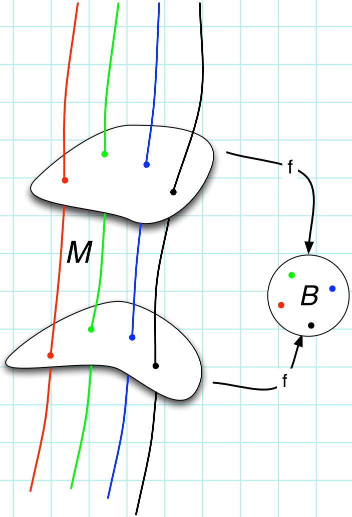
- $F_a^A := \nabla_a f^A$ has maximal rank
- $F_a^A : TM \to TB$ is a two-point tensor
- one-dimensional kernel, time-like

$$u^a F_a^A = 0, \qquad u^a u_a = 1$$

- 4-velocity of individual particles

Geometry and Kinematics

Strain



- fix a foliation of M (time coordinate t)
- $f : \{t = const.\} \to B$ local diffeo
- push-forward $-f_*(g^{-1})$ is a family of positive (inverse) metrics on B

$$H^{AB}(t) := -F_a^A F_b^B g^{ab} = -F_a^A F_b^B h^{ab}$$

- current spatial distance between particles
- state of strain

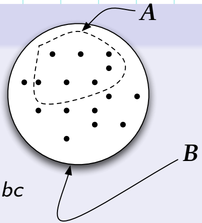
Geometry and Kinematics

Material properties

Material properties require specific, pre-existing structures on B

examples:

1. volume structure, 3-form η_{ABC} on B



- $\int_A \eta$ is number of particles inside $A \subset B$
- pullback $f^*\eta$ is 3-form on M

$$\eta_{abc} = F_a^A F_b^B F_c^C \eta_{ABC} = n\, u^d \epsilon_{dabc}$$

- particle density (concentration) n, mass density $\rho = mn$
- material current $j^a = nu^a$, 'baryon' conservation

$$\nabla_a j^a = 0$$

Geometry and Kinematics

Material properties

Material properties require specific, pre-existing structures on B

examples:

2. reference state G_{AB} for the strain

- metric on B (flat?)
- defines a relaxed state
- discrepancy between H^{AB} and G^{AB} is deformation,

$$\epsilon_{AB} = H_{AB} - G_{AB}, \quad \text{or } \epsilon = -\frac{1}{2}\log(HG^{-1})$$

Other structures:

3. high-pressure formulation requires conformal metric
4. anisotropy properties require vector fields, etc.

Dynamics

Action principle with the energy density ρ

$$\mathcal{A} = \int \rho[f, \nabla f; g] \sqrt{-g}\, d^4x$$

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Variation with respect to f: Euler-Lagrange equations

$$E_A \equiv \frac{1}{\sqrt{-g}} \nabla_a \left[\sqrt{-g} \frac{\partial \rho}{\partial F_a^A} \right] - \frac{\partial \rho}{\partial f^A} = 0$$

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divergence of energy-momentum tensor

$$\nabla^b T_{ba} = E_A F_a^A$$

material equations are satisfied $\iff$ emt is divergence free

Dynamics

introduce energy-per-particle: $e = \rho/n$

(2^{nd}) Piola-Kirchhoff stress tensor

$$\tau_{AB} = 2\frac{\partial e}{\partial H^{AB}}$$

Energy-momentum tensor takes the form

$$T_{ab} = \rho\, u_a u_b - F_a^A F_b^B \tau_{AB}$$

compare with ideal fluid $T_{ab} = \rho\, u_a u_b - p h_{ab}$ (obtained for $f = f(n)$).

Dynamics

Hooke approximation: stress is proportional to strain

$$\tau^{AB} \propto \epsilon^{AB}$$

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$$e = m + \frac{m}{2} K_{ABCD} \epsilon^{AB} \epsilon^{CD} + \mathcal{O}(\epsilon^3)$$

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for isotropic materials with Lamé coefficients λ and μ

$$e = m + \frac{m}{4} \left(2\lambda (\epsilon_A{}^A)^2 + \mu \epsilon_{AB} \epsilon^{AB} \right)$$

Spherically symmetric and static systems

Purpose: check the differences when choosing

- different relaxed states (irrelevant in the fluid case)
- different matter models
- different strain variables

Spherically symmetric and static systems

reference metric on B, spherically symmetric

- flat euclidean metric (why?)
- inner Schwarzschild solution

energy density

$$\rho = \rho_0 \left(1 + \frac{1}{4}\left(2\lambda(\mathrm{Tr}\,\epsilon)^2 + \mu \mathrm{Tr}(\epsilon^2)\right)\right)$$

matter models (fixed by choice of ρ_0, λ, μ)

- elastic aluminium sphere
  $\rho_0 = 2.72$ g/cm^3, $\lambda = 63.3$ GPa, $\mu = 25$ GPa.
- sphere made from nucleonic matter
  $\rho_0 = 10^{14}$ g/cm^3, $\lambda = 10^{20}$ GPa, $\mu = 10^{17}$ GPa.

Spherically symmetric and static systems

spherically symmetric space-time metric

$$g = e^{2\eta}\, dt^2 - e^{2\xi}\, dr^2 - r^2\, (d\theta^2 + \sin^2\theta\, d\phi^2)$$

Spherically symmetric and static systems

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spherically symmetric body metric

$$G = e^{2\xi_0}\, dr^2 + r^2\,(d\theta^2 + \sin^2\theta\, d\phi^2)$$

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equivariant map f

$$f(t, r, \theta, \phi) = (R(r), \theta, \phi)$$

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deformation tensor for BS ($x = R' e^{-(\xi - \xi_0)}$ and $y = R/r$)

$$\epsilon_a{}^b \doteq (x^2 - 1) dr \otimes \partial_r + (y^2 - 1)\,(d\theta \otimes \partial_\theta + d\phi \otimes \partial_\phi)$$

deformation is determined by x and y, KM below

Einstein equations

$$G_0{}^0 = e^{-2\xi}\left\{\frac{1}{r^2}\left(1-e^{2\xi}\right)-\frac{2}{r}\xi'\right\} = -8\pi\,\rho,$$

$$G_1{}^1 = e^{-2\xi}\left\{\frac{1}{r^2}\left(1-e^{2\xi}\right)+\frac{2}{r}\eta'\right\} = 8\pi\,P,$$

$$G_2{}^2 = e^{-2\xi}\left\{\eta''+(\eta')^2-\eta'\xi'-\frac{1}{r}\left(\eta'-\xi'\right)\right\} = 8\pi\,Q,$$

with $T^0{}_0 = \rho$, $P = -T^1{}_1$, $Q = -T^2{}_2$

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with $T^0{}_0 = \rho,\ P = -T^1{}_1,\ Q = -T^2{}_2$

divergence-free condition (Bianchi identity)

$$P' + \eta'(\rho + P) + \frac{2}{r}(P - Q) = 0$$

new variable: ‘averaged’ energy density

$$w = \frac{4\pi}{r^3}\int_0^r \rho(s)s^2\,ds$$

new variable: 'averaged' energy density

$$w = \frac{4\pi}{r^3}\int_0^r \rho(s)s^2\,ds$$

new equations

$$\begin{aligned} rw' &= -3w + 4\pi\rho, \\ r\eta' &= r^2\frac{4\pi P + w}{1 - 2r^2 w}, \\ rP' &= -r^2\frac{4\pi P + w}{1 - 2r^2 w}(\rho + P) - 2(P - Q). \end{aligned}$$

Clearly, $P = P(x, y)$ and $Q = Q(x, y)$

$$P' = P_x x' + P_y y'$$

final system:

$$rw' = -3w + 4\pi\rho,$$

$$ry' = \frac{x}{\sqrt{1-2r^2w}} e^{-\xi_0} - y,$$

$$rx' = \frac{P_y}{P_x}\left(y - \frac{x}{\sqrt{1-2r^2w}} e^{-\xi_0}\right) - r^2 \frac{4\pi P + w}{1-2r^2w} \frac{\rho + P}{P_x} - \frac{2}{P_x}(P - Q)$$

- singular at $r = 0$, existence of solutions must be checked
- regularity requires special initial conditions

$$w(0) = \frac{4\pi}{3}\rho(0), \quad y(0) = x(0) =: a, \quad P(0) = Q(0)$$

- can be solved numerically

conventional deformation variable (BS)

$$\rho = xy^2 \left(\rho_0 + \frac{\mu}{4} \left\{ (x^2 - 1)^2 + 2(y^2 - 1)^2 \right\} + \frac{\lambda}{8} \left\{ (x^2 - 1) + 2(y^2 - 1) \right\}^2 \right),$$

$$P = \frac{1}{2}(xy^2)x^2 \left((\lambda + 2\mu)(x^2 - 1) + 2\lambda(y^2 - 1) \right),$$

$$Q = \frac{1}{2}(xy^2)y^2 \left((2\lambda + 2\mu)(y^2 - 1) + \lambda(x^2 - 1) \right).$$

logarithmic deformation variable (KM)

$$\rho = xy^2 \left(\rho_0 + \mu \left\{ (\log x)^2 + 2(\log y)^2 \right\} + \frac{\lambda}{2} \left\{ \log x + 2 \log y \right\}^2 \right),$$

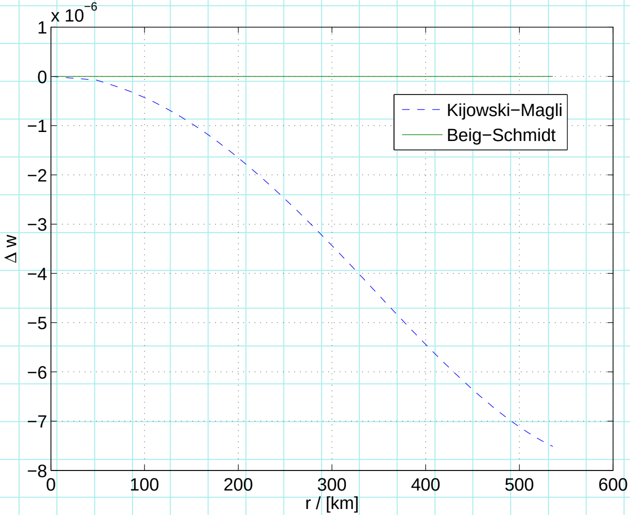
$$P = (xy^2) \left((\lambda + 2\mu) \log x + 2\lambda \log y \right),$$

$$Q = (xy^2) \left((2\lambda + 2\mu) \log y + \lambda \log x \right).$$

Results

Small deformations, aluminum sphere, differences between classical and relativistic

- aluminium sphere
- for classical elasticity, BS and KM energy functional
- relative central compression $\delta = 10^{-3}$
- radius 535 km, mass 1.2×10^{12} kg
- plot: relative difference in mean density to classical solution


x 10^−6
1
0
−1
−2
−3
−4
−5
−6
−7
−8
Δ w
0
100
200
300
400
500
600
r / [km]
Kijowski−Magli
Beig−Schmidt

Results

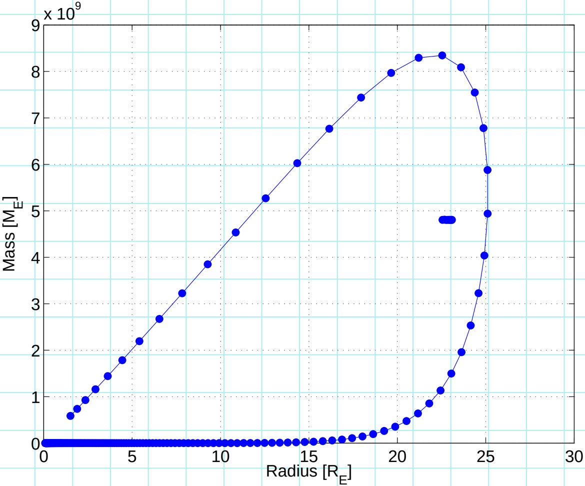
Small deformations, aluminum sphere, differences between classical and relativistic

- aluminium sphere
- for classical elasticity, BS and KM energy functional
- relative central compression $\delta = 1/1000$
- radius 535 km, mass 1.2×10^{12} kg
- difference between BS and classical solution $\leq 6 \times 10^{-13}$
- KM is more tightly bound than BS and classical solution

Results

Aluminum, large deformations, BS energy

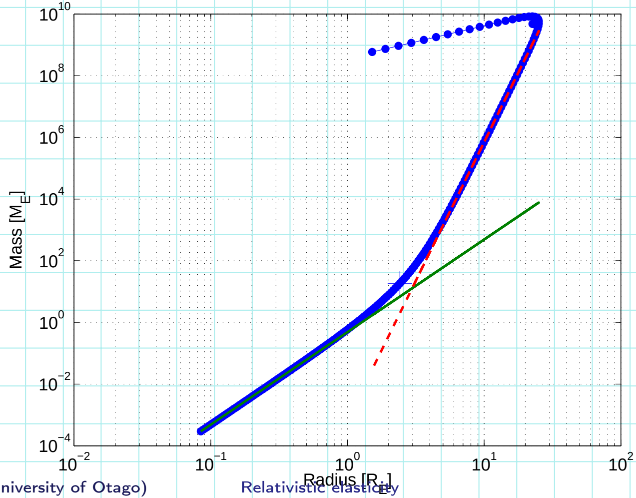
mass-radius diagram (linear axes)



Results

Aluminum, large deformations, BS energy

mass-radius diagram (double logarithmic axes)



Results

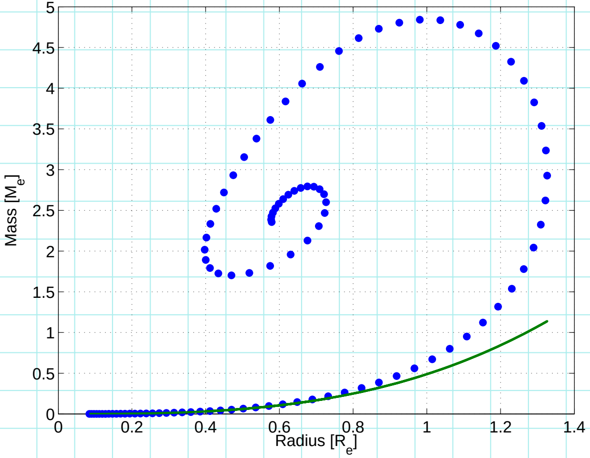
Aluminum, large deformations, BS energy

- maximal mass $\sim 8.4 \times 10^9 M_E$
- maximal radius $\sim 25 R_E$
- three different regimes
- classical $M \propto R^3$
- extreme $M \propto R^9$
- linear unstable $M \propto R$
- second (stable?) branch

Results

Aluminum, large deformations, KM energy

mass-radius diagram (linear axes)



Results

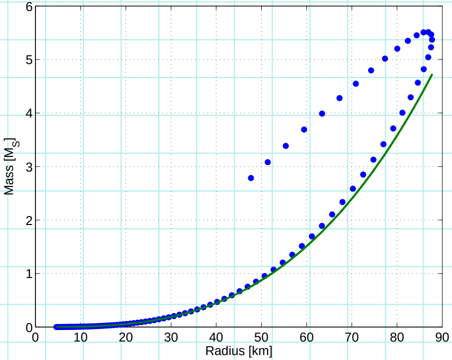
Aluminum, large deformations, KM energy

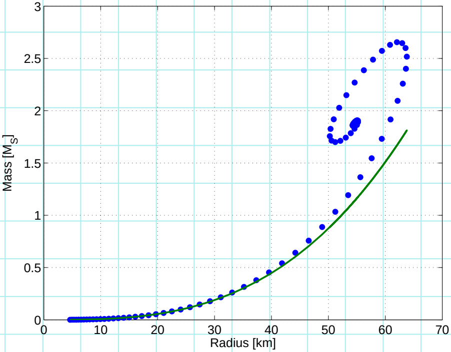
- maximal mass $\sim 5M_E$
- maximal radius $\sim 1.3R_E$
- classical regime $M \propto R^3$
- much more compact configurations
- convergence to limit configuration

Results

Nucleonic matter, large deformations

mass-radius diagram (linear axes)





Conclusions

- families of spherical elastic bodies
- large deformations, for small ones no relevant differences
- KM energy forces convergence to a limit configuration
- BS energy shows three different, very distinct regimes
- and possibly more than one branch in mass-radius diagram
- difference in relaxed state generally negligible
- no qualitative difference between aluminium and nucleonic matter
- proceed to do perturbations coupled to gw