Conformally related material metrics in General Relativistic Elasticity

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1. General relativistic elasticity
2. Conformal material metrics
3. Applications to static spherically symmetric configurations

Basic concepts

Configuration mapping

The space-time configuration of the material is described by the mapping

$$\Psi : M \longrightarrow X.$$

- (M, g_{ab}) space-time with coordinate system $\{\omega^a\}$, $a = 0, 1, 2, 3$
- (X, K_{AB}) material space with material metric K_{AB} and coordinate system $\{\xi^A\}$, $A = 1, 2, 3$

Basic concepts

Pulled-back material metric

$$k_{ab} = \Psi^* K_{AB} = \xi^A_a \xi^B_b K_{AB},$$

where $\xi^A_a = \frac{\partial \xi^A}{\partial \omega^a}$ is the relativistic deformation gradient.

Let n_1^2, n_2^2, n_3^2 be the eigenvalues of k^a_b:

$$k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b$$

Basic concepts

Orthonormal tetrad $\{u^a, x^a, y^a, z^a\} = \{e^a_\mu\}$

$\mu, \nu, \rho ... = 0, 1, 2, 3$
$\alpha, \beta, \gamma ... = 1, 2, 3$

x, y, z are the eigenvectors of k^a_b.

u is the velocity field of the matter: $u^a \xi^A_a = 0$, $u^a u_a = -1$, $u^0 > 0$.

$$g_{ab} = -u_a u_b + x_a x_b + y_a y_b + z_a z_b$$

$h_{ab} = x_a x_b + y_a y_b + z_a z_b$ projection tensor

Basic concepts

Relativistic strain tensor

$$s_{ab} = \frac{1}{2}(h_{ab} - k_{ab})$$

The material is in an unstrained state if $s_{ab} = 0$.

Basic concepts

Energy-momentum tensor

$$T_{ab} = -\rho g_{ab} + 2\frac{\partial \rho}{\partial g^{ab}} = \rho u_a u_b + p_{ab}.$$

- $p_{ab} = 2\frac{\partial \rho}{\partial g^{ab}} - \rho h_{ab}$ pressure tensor
- $\rho = n\epsilon$ energy density
- ϵ energy per particle
- n particle number density

Basic concepts

Elasticity Difference Tensor

For any vector field X, one has

$$\tilde{D}_b X^a - D_b X^a = h_b^m h_n^a (\tilde{\nabla}_m X^n - \nabla_m X^n) = S^a{}_{bc} X^c,$$

where the elasticity difference tensor $S^a{}_{bc}$ can be expressed as

$$S^a{}_{bc} = \frac{1}{2} k^{-am} (D_b k_{mc} + D_c k_{mb} - D_m k_{bc}),$$

where k^{-am} is such that $k^{-am} k_{mb} = h_b^a$.

Decomposition for the EDT

Decomposing ${S^a}_{bc}$ along the eigenvectors of k^a_b one obtains

$$ {S^a}_{bc} = \underset{1}{M}_{bc} x^a + \underset{2}{M}_{bc} y^a + \underset{3}{M}_{bc} z^a . $$

$\underset{1}{M}_{bc}$, $\underset{2}{M}_{bc}$ and $\underset{3}{M}_{bc}$ are second order symmetric, flowline orthogonal tensors.

Conformal material metrics

Consider two conformally related pulled-back material metrics defined on (M, g) obtained from two conformal metrics $K_{AB} = f^2 \bar{K}_{AB}$ on X:

$$k_{ab} = f^2 \bar{k}_{ab},$$

where $f = f(\xi^A(w^a))$ is a smooth, strictly positive function.

- $k_{ab} = n_1^2 x_a x_b + n_2^2 y_a y_b + n_3^2 z_a z_b$
- $\bar{k}_{ab} = \bar{n}_1^2 x_a x_b + \bar{n}_2^2 y_a y_b + \bar{n}_3^2 z_a z_b$

Consequences

Eigenvalues of the pulled-back material metrics

- $n_1^2 = f^2 \bar{n}_1^2$
- $n_2^2 = f^2 \bar{n}_2^2$
- $n_3^2 = f^2 \bar{n}_3^2$

Particle number densities

- $n = f^3 \bar{n}$

Consequences

Energy-momentum tensor

For k: $$T_{ab} = n\epsilon u_a u_b + n n_1 \frac{\partial\epsilon}{\partial n_1} x_a x_b + n n_2 \frac{\partial\epsilon}{\partial n_2} y_a y_b + n n_3 \frac{\partial\epsilon}{\partial n_3} z_a z_b$$

For $\bar{k}$: $$\bar{T}_{ab} = \bar{n}\bar{\epsilon} u_a u_b + \bar{n}\bar{n}_1 \frac{\partial\bar{\epsilon}}{\partial \bar{n}_1} x_a x_b + \bar{n}\bar{n}_2 \frac{\partial\bar{\epsilon}}{\partial \bar{n}_2} y_a y_b + \bar{n}\bar{n}_3 \frac{\partial\bar{\epsilon}}{\partial \bar{n}_3} z_a z_b$$

$T_{ab} = \bar{T}_{ab} \Rightarrow$

Energy per particle

- $\epsilon = \frac{1}{f^3}\bar{\epsilon}$
- $\frac{\partial\epsilon}{\partial n_1} = \frac{1}{f^4}\frac{\partial\bar{\epsilon}}{\partial \bar{n}_1}, \quad \frac{\partial\epsilon}{\partial n_2} = \frac{1}{f^4}\frac{\partial\bar{\epsilon}}{\partial \bar{n}_2}, \quad \frac{\partial\epsilon}{\partial n_3} = \frac{1}{f^4}\frac{\partial\bar{\epsilon}}{\partial \bar{n}_3}$

Consequences

Energy densities

$$\rho = \bar{\rho}$$

Strain tensors

$$s_{ab} = f^2 \bar{s}_{ab} + \frac{1}{2} h_{ab} (1 - f^2)$$

Elasticity difference tensor

Relation between ${S^a}_{bc}$ and ${\bar{S}^a}_{bc}$

$$ {S^a}_{bc} = {\bar{S}^a}_{bc} + \frac{1}{f}\left(h^a_c\, D_b f + h^a_b\, D_c f - \bar{k}^{-am}\bar{k}_{bc}\, D_m f\right), $$

where $\bar{k}^{-am}\bar{k}_{bc} = k^{-am}k_{bc}$

Decomposition for the Elasticity Difference Tensor

Relation between $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$

$$\underset{\alpha}{M}_{bc} = \underset{\alpha}{\bar{M}}_{bc} + \frac{1}{f}\left(e_c^{\alpha} D_b f + e_b^{\alpha} D_c f - \frac{1}{n_{\alpha}^2} k_{bc} e^{\alpha m} D_m f \right)$$

Eigenvalue-eigenvector problem

Eigenvalue-eigenvector equation for $\underset{\alpha}{M}$

$$\underset{\alpha}{M}{}_b^c \omega^b = \lambda \omega^c$$

Eigenvalue-eigenvector equation for $\underset{\alpha}{\bar{M}}$

$$\underset{\alpha}{\bar{M}}{}_b^c \omega^b = \bar{\lambda} \omega^c$$

Investigate the conditions for the eigenvectors of the pulled-back material metric $\omega^b = x^b$, $\omega^b = y^b$, $\omega^b = z^b$ to be eigenvectors for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$.

Eigenvalue-eigenvector problem

Conditions for e_α to be an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$

- $\Delta_{e_\beta}(\ln f) = 0$ for each $\beta \neq \alpha$

Eigenvalues

- $\lambda = \bar{\lambda} + \Delta_{e_\alpha}(\ln f)$

$$\Delta_{e_\alpha}(\ln f) = (\ln f)_{,m} e_\alpha^m$$

Conditions for e_β to be an eigenvector for $\underset{\alpha}{M}$ and $\underset{\alpha}{\bar{M}}$

- $\Delta_{e_\beta}(\ln f) = 0$, for a fixed $\beta \neq \alpha$
- $\underset{\alpha}{\bar{M}}{}_b^c e_\beta^b e_{\gamma c} = 0$

Eigenvalues

- $\lambda = \bar{\lambda} - \frac{n_\beta^2}{n_\alpha^2} \Delta_{e_\alpha}(\ln f)$

Static spherically symmetric space-time

Space-time (M, g)

- Static spherically symmetric metric

$$ds^2 = -e^{2\nu(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

- Coordinates: $\omega^a = \{t, r, \theta, \phi\}$

Static spherically symmetric space-time

Pulled-back material metrics

1. $\bar{k}_{ab}$ $\quad ds^2 = \tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2$

2. k_{ab} $\quad ds^2 = f^2(\tilde{r})[\tilde{r}'^2\, dr^2 + \tilde{r}^2\, d\theta^2 + \tilde{r}^2\, sin^2\theta\, d\phi^2]$

Coordinates on X:

$$\xi^A = \{\tilde{r}, \tilde{\theta}, \tilde{\phi}\}, \;\; \tilde{r} = \tilde{r}(r), \;\; \tilde{\theta} = \theta, \;\; \tilde{\phi} = \phi$$

Static spherically symmetric space-time

Eigenvalues of $\bar{k}^a_b$

$$\bar{n}_1^2 = \tilde{r}'^2\, e^{-2\lambda}$$

$$\bar{n}_2^2 = \bar{n}_3^2 = \frac{\tilde{r}^2}{r^2}$$

Eigenvalues of k^a_b

$$n_1^2 = f^2 \tilde{r}'^2\, e^{-2\lambda}$$

$$n_2^2 = n_3^2 = f^2 \frac{\tilde{r}^2}{r^2}$$

Static spherically symmetric space-time

Strain tensor associated with $\bar{k}^a_b$

$$\bar{s}_{rr} = \frac{1}{2}\left(e^{2\lambda} - \tilde{r}'^2\right) \qquad \bar{s}_{\theta\theta} = \frac{1}{2}\left(r^2 - \tilde{r}^2\right) \qquad \bar{s}_{\phi\phi} = \frac{1}{2}\sin^2\theta\left(r^2 - \tilde{r}^2\right)$$

Strain tensor associated with k^a_b

$$s_{rr} = \frac{1}{2}\left(e^{2\lambda} - f^2\tilde{r}'^2\right) \quad s_{\theta\theta} = \frac{1}{2}\left(r^2 - f^2\tilde{r}^2\right) \quad s_{\phi\phi} = \frac{1}{2}\sin^2\theta\left(r^2 - f^2\tilde{r}^2\right)$$

$\bar{s}_{ab}$ and s_{ab} vanish iff $\tilde{r}(r) = ce^{\pm\int \frac{e^\lambda}{r}dr}, \; c > 0.$

Elasticity Difference Tensor

For $\bar{k}$

$$\bar{S}^{r}{}_{rr} = \frac{\tilde{r}''}{\tilde{r}'} - \lambda'$$

$$\bar{S}^{\theta}{}_{\theta r} = \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r}$$

$$\bar{S}^{\phi}{}_{\phi r} = \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r}$$

$$\bar{S}^{r}{}_{\theta\theta} = re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}$$

$$\bar{S}^{r}{}_{\phi\phi} = sin^2\theta \left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} \right)$$

Elasticity Difference Tensor

For k

$$S^r{}_{rr} = \frac{\tilde{r}''}{\tilde{r}'} - \lambda' + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'$$

$$S^\theta{}_{\theta r} = \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'$$

$$S^\phi{}_{\phi r} = \frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'$$

$$S^r{}_{\theta\theta} = re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{\tilde{r}^2}{\tilde{r}'}\frac{1}{f}\frac{df}{d\tilde{r}}$$

$$S^r{}_{\phi\phi} = sin^2\theta\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{\tilde{r}^2}{\tilde{r}'}\frac{1}{f}\frac{df}{d\tilde{r}}\right)$$

1. Eigenvectors and eigenvalues for $\underset{1}{\bar{M}}$, $\underset{2}{\bar{M}}$ and $\underset{3}{\bar{M}}$

	Eigenvectors	Eigenvalues
$\underset{1}{\bar{M}}$	x	$\mu_1 = e^{-\lambda}\left(\frac{\tilde{r}''}{\tilde{r}'} - \lambda'\right)$
	y	$\mu_2 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$
	z	$\mu_3 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'}\right)$
$\underset{2}{\bar{M}}$	$x+y$	$\mu_4 = e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
	$x-y$	$\mu_5 = -e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
	z	$\mu_6 = 0$
$\underset{3}{\bar{M}}$	$x+z$	$\mu_7 = e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
	$x-z$	$\mu_8 = -e^{-\lambda}(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r})$
	y	$\mu_9 = 0$

2. Eigenvectors and eigenvalues for $\underset{1}{M}$, $\underset{2}{M}$ and $\underset{3}{M}$

	Eigenvectors	Eigenvalues
$\underset{1}{M}$	x	$\mu_1 = e^{-\lambda}\left(\frac{\tilde{r}''}{\tilde{r}'} - \lambda' + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
	y	$\mu_2 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{1}{f}\frac{df}{d\tilde{r}}\frac{\tilde{r}^2}{\tilde{r}'}\right)$
	z	$\mu_3 = \frac{e^{\lambda}}{r^2}\left(re^{-2\lambda} - \frac{\tilde{r}}{\tilde{r}'} - \frac{1}{f}\frac{df}{d\tilde{r}}\frac{\tilde{r}^2}{\tilde{r}'}\right)$
$\underset{2}{M}$	$x+y$	$\mu_4 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
	$x-y$	$\mu_5 = -e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
	z	$\mu_6 = 0$
$\underset{3}{M}$	$x+z$	$\mu_7 = e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
	$x-z$	$\mu_8 = -e^{-\lambda}\left(\frac{\tilde{r}'}{\tilde{r}} - \frac{1}{r} + \frac{1}{f}\frac{df}{d\tilde{r}}\tilde{r}'\right)$
	y	$\mu_9 = 0$

References

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