Spinor calculus on 5-dimensional spacetimes

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Spinors in a 5-dimensional Lorentzian vector space

Spinor calculus in a 5-dimensional Lorentzian manifold

Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

Conclusions

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Outline

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2. Spinor calculus in a 5-dimensional Lorentzian manifold
   - The spin covariant derivative
   - The curvature spinors
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3. Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

4. Conclusions
The spacetime tensor algebra

Let $\mathbf{L}$ be a 5-dimensional real vector space endowed with a real scalar product $g(\ ,\ )$ of Lorentzian signature. Use the vector space $\mathbf{L}$ and its dual $\mathbf{L}^*$ to build a tensor algebra $\mathcal{T}(\mathbf{L})$.

The spinorial tensor algebra

Let $\mathbf{S}$ be a complex vector space whose dimension is for the moment left unspecified. Use the vector space $\mathbf{S}$ and its dual $\mathbf{S}^*$ to build a tensor algebra $\mathcal{T}(\mathbf{S})$.

Notation

We use abstract indices to denote tensorial quantities: small Latin letters $a, b, \ldots$ are indices for elements in $\mathcal{T}(\mathbf{L})$ and uppercase Latin letters $A, B, \ldots$ are indices for elements in $\mathcal{T}(\mathbf{S})$. 
We introduce now a mixed quantity $\gamma_{aB}^C$ which fulfils the following algebraic property

$$\gamma_{aA}^B \gamma_{bB}^C + \gamma_{bA}^B \gamma_{aB}^C = -\delta_{A}^{C} g_{ab},$$

This relation means that $\gamma_{aB}^C$ can be regarded as belonging to a representation on the vector space $S$ of the Clifford algebra $Cl(L, g)$. 
If the quantity $\gamma_{aB}^C$ belongs to an irreducible representation of $\text{Cl}(\mathbf{L}, g)$, then:

- the dimension of $\mathbf{S}$ is 4.
- There exist two antisymmetric spinors $\epsilon_{AB}$, $\hat{\epsilon}^{AB}$, unique up to a constant, such that

$$\epsilon_{AB} \hat{\epsilon}^{CB} = \delta_A^C,$$

$$\gamma_{aD}^A \gamma^a_{C} B = \frac{1}{2} \delta_{D}^A \delta_{C}^B - \delta_{C}^A \delta_{D}^B + \epsilon_{CD} \hat{\epsilon}^{AB}.$$
The quantities $\hat{\epsilon}^{AB}$, $\epsilon_{AB}$ can be used to raise and lower spinor indices. Example:

$$\xi^A \epsilon_{AB} = \xi_B, \quad \xi^A = \hat{\epsilon}^{AB} \xi_B.$$ 

From now on we set $\hat{\epsilon}^{AB} = \epsilon^{AB}$.

Previous theorem entails the relations

$$\gamma_a{}^{AB} \gamma_b{}^{AB} = -2g_{ab}, \quad \gamma^a{}^{[AB]} = \gamma^a{}_{AB}, \quad \epsilon^{AB} \gamma^a{}_{AB} = 0,$$

$$\gamma_a{}^{CD} \gamma^a{}_{AB} = \epsilon^{AD} \epsilon^{BC} - \epsilon^{AC} \epsilon^{BD} + \frac{1}{2} \epsilon^{AB} \epsilon^{CD}.$$ 

Use these properties to transform tensors into spinors and back. Example:

$$v^{AB} = \gamma_a{}^{AB} v^a, \quad v^a = -\frac{1}{2} \gamma^a{}_{AB} v^{AB}.$$ 

If $v^{AB}$ is given, then $v^a$ is completely determined. However, for a given $v^a$ the spinor $v^{AB}$ is not determined unless $v^{AB}$ is antisymmetric and trace-free.
The spin tetrad and the semi-null pentad

Let \( \{o^A, \iota^A, \tilde{o}^A, \tilde{\iota}^A\} \) be a basis in \( S \) such that

\[
\epsilon_{AB} = 2o_{[A}\iota_{B]} + 2\tilde{o}_{[A}\tilde{\iota}_{B]}.
\]

This entails

\[
o^A\iota_A = -1 = \tilde{o}^A\tilde{\iota}_A, \quad o^A\tilde{o}_A = \iota^A\tilde{\iota}_A = o^A\tilde{\iota}_A = \iota^A\tilde{o}_A = 0
\]

The basis \( \{o^A, \iota^A, \tilde{o}^A, \tilde{\iota}^A\} \) is a spin tetrad. From it we construct a semi-null pentad as follows

\[
\begin{align*}
l^a &\equiv \gamma^a_{AB}o^A\tilde{o}^B, & n^a &\equiv \gamma^a_{AB}\iota^A\tilde{\iota}^B, \\
m^a &\equiv -o^A\gamma^a_{AB}\iota^B, & \bar{m}^a &\equiv \tilde{o}^A\gamma^a_{AB}\iota^B, \\
u^a &\equiv 2o^B\gamma^a_{AB}\iota^A = -2\tilde{o}^B\gamma^a_{AB}\tilde{\iota}^A.
\end{align*}
\]

Hence

\[
l^a n_a = -1, \quad m^a \bar{m}_a = 1, \quad u^a u_a = -2.
\]
Spin structures on a 5-dimensional Lorentzian manifold

Let \((\mathcal{M}, g)\) be a 5-dimensional Lorentzian manifold and let \(T_p(\mathcal{M})\) be the tangent space at a point \(p\). We can introduce a spin space \(S_p\) and the quantity \(\gamma_{aA}^B|_p\) at each point \(p\).

**Definition: Spin bundle**

The union

\[
S(\mathcal{M}) \equiv \bigcup_{p \in \mathcal{M}} S_p,
\]

is a vector bundle called the spin bundle. The sections of \(S(\mathcal{M})\) are the contravariant rank-1 spinor fields on \(\mathcal{M}\).

We can define tensor bundles with the tensor products of space-time tensors and spinors as building blocks. We use \(\mathcal{G}(\mathcal{M})\) as the generic symbol to denote these tensor bundles.
Definition: spin structure on a 5-dimensional manifold

If the quantity $\gamma_{aA}^B|_p$ varies smoothly on the manifold $\mathcal{M}$, then one can define a smooth section of the bundle $\mathcal{S}^{0,1}_{1,1}(\mathcal{M})$, denoted by $\gamma_{aA}^B$. When this is the case we call the smooth section $\gamma_{aA}^B$ a smooth spin structure on the Lorentzian manifold $(\mathcal{M}, g)$.

As a consequence, we can now define two smooth sections $\epsilon_{AB}, \epsilon^{AB}$ of $\mathcal{S}(\mathcal{M})$ and use them to lower and raise spinor indices, just as we did before.
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The spin covariant derivative

Definition: (Spin covariant derivative)

Suppose that $\mathcal{S}(\mathcal{M})$ admits a spin structure $\gamma^B_{aA}$. We say that a covariant derivative $D_a$ defined on $\mathcal{S}(\mathcal{M})$ is compatible with the spin structure $\gamma^B_{aA}$ if

$$D_a \gamma^D_{bC} = 0.$$ 

The covariant derivative $D_a$ is then called a spin covariant derivative with respect to the spin structure $\gamma^B_{aA}$.

Elementary properties of a spin covariant derivative are

$$D_a g_{bc} = 0,$$

$$D_a \epsilon_{AB} = \frac{1}{4} (\epsilon^{CD} D_a \epsilon_{CD}) \epsilon_{AB}.$$
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Theorem

There is one and only one spin covariant derivative $\nabla_a$ on $\mathcal{S}(\mathcal{M})$ with respect to the spin structure $\gamma_{aA}^B$ which fulfils the property

$$\nabla_a \epsilon_{AB} = 0.$$  

From now on we will restrict ourselves to $\nabla_a$. The spinor form of this operator is

$$\nabla_{AB} \equiv \gamma^a_{AB} \nabla_a \Rightarrow \nabla_{[AB]} = \nabla_{AB} \ , \ \nabla^A_A = 0$$
The curvature spinors

The Riemann tensor \( R_{abcf} \) of the covariant derivative \( \nabla_a \) can be decomposed in the form

\[
R_{abcf} = \Lambda (g_{af}g_{bc} - g_{ac}g_{bf}) - \frac{1}{2} G_{ab}^{AB} G_{cf}^{CD} \Omega_{ABCD} - G_{ab}^{AB} G_{cf}^{CD} \Psi_{ACBD},
\]

where

\[
G_{ab}^{AC} \equiv -\gamma^a (A^B \gamma^b C) B.
\]

The quantities \( \Lambda \), \( \Omega_{ABCD} \) and \( \Psi_{ABCD} \) are known collectively as the curvature spinors.
Algebraic properties of the curvature spinors

\[ \Psi_{ABCD} \equiv \text{Weyl spinor}, \quad \Omega_{ABCD} \equiv \text{Ricci spinor}, \quad \Lambda \equiv \text{scalar curvature}. \]

\[ \Psi_{(ABCD)} = \Psi_{ABCD} \]

\[ \Psi_{ABCD} \text{ has 35 independent components.} \]

\[ \Omega_{ABCD} = \Omega_{[AB]CD} = \Omega_{CDAB}, \quad \Omega_{AB}^C{}^C = \Omega_A^C{}^C{}^D = 0, \]
\[ \Omega_{ABCD} + \Omega_{BCAD} + \Omega_{CABD} = 0. \]

\[ \Omega_{ABCD} \text{ has 14 independent components.} \]

\[ 35 + 14 + 1 = 50, \quad \text{Number of independent components of } R_{abcd}. \]
Bianchi and Ricci identities

Proposition: spinor form of the second Bianchi identity

The curvature spinors fulfill the differential identity

\[ \nabla_{(Z^W \Psi^V)} B A W - \nabla_{(Z^W \Omega^V)} A B W - \nabla_{(Z^W \Omega^V)} B A W - 2 \epsilon(A | (V \nabla Z)| B) \Lambda = 0. \]

We introduce the linear differential operator

\[ \Box_{AB} \equiv G^{ab}_{\ AB} \nabla_a \nabla_b \Rightarrow \Box_{(AB)} = \Box_{AB}. \]

Proposition: spinor form of the Ricci identity

For any spinor \( \xi_B \) we have

\[ \Box_{CD} \xi_B = \Lambda \epsilon_B(C \xi_D) - \xi^A \Omega_B(CD)A - \frac{1}{2} \xi^A \Psi_{BCDA}. \]

The action of \( \Box_{CD} \) can be extended to any spinor by means of the linearity and the Leibnitz rule.
Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- The frame derivations:
  \[
  D \equiv l^a \nabla_a, \quad \Delta \equiv n^a \nabla_a, \quad \delta \equiv m^a \nabla_a, \quad \bar{\delta} \equiv \bar{m}^a \nabla_a, \quad D \equiv u^a \nabla_a.
  \]

- The spin coefficients:
  \[
  D_0 A = \epsilon o_A + \bar{\eta} i_A - \kappa \iota_A + \chi \tilde{\o}_A, \quad \Delta o_A = \gamma o_A + \bar{\epsilon} A - \tau \iota_A + \omega \tilde{\o}_A, \\
  \delta o_A = \beta o_A - \sigma \iota_A + \chi \tilde{\o}_A + \phi \tilde{\o}_A, \quad \bar{\delta} o_A = \alpha o_A - \mu \iota_A + \zeta \tilde{\o}_A + \eta \tilde{\o}_A, \\
  D_0 A = -\theta o_A + \eta \iota_A + \bar{\xi} \iota_A + \psi \tilde{\o}_A, \quad D \iota_A = \zeta \tilde{\o}_A - \epsilon \iota_A + \pi \tilde{\o}_A - \bar{\chi} \iota_A, \\
  \Delta \iota_A = \beta o_A - \gamma \iota_A + \nu \tilde{\o}_A - \bar{\omega} \tilde{\o}_A, \quad \delta \iota_A = -\beta \iota_A + \mu \tilde{\o}_A + \xi \tilde{\o}_A - \bar{\psi} \iota_A, \\
  \bar{\delta} \iota_A = -\alpha \iota_A + \lambda \tilde{\o}_A + \xi \tilde{\o}_A - \bar{\phi} \iota_A, \quad D \iota_A = \xi \tilde{\o}_A + \zeta \tilde{\o}_A + \theta \tilde{\o}_A - \bar{\psi} \iota_A.
  \]

Twelve Newman-Penrose 4-D spin coefficients
\[
\alpha, \, \beta, \, \gamma, \, \epsilon, \, \kappa, \, \lambda, \, \mu, \, \nu, \, \pi, \, \rho, \, \sigma, \, \tau.
\]

Ten complex 5-D spin coefficients
\[
\zeta, \, \eta, \, \theta, \, \chi, \, \omega, \, \phi, \, \xi, \, \nu, \, \psi, \, \varsigma.
\]

Six real 5-D spin coefficients
\[
a, \, b, \, c, \, \bar{d}, \, e, \, \bar{f}.
\]

\[2 \times 12 + 2 \times 10 + 6 = 50\] real Ricci rotation coefficients.
Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

Components of the Weyl spinor:

\[ \Psi_0 \equiv \Psi_{ABCD} o^A o^B o^C o^D, \quad *\Psi_0 \equiv \Psi_{ABCD} o^A o^B o^C o^D, \]

\[ \Psi_1 \equiv \Psi_{ABCD} o^A o^B o^C i^D, \quad *\Psi_1 \equiv \Psi_{ABCD} o^A o^B c^C i^D, \quad \Psi_1^* \equiv \Psi_{ABCD} o^A o^B c^C i^D, \]

\[ \Psi_2 \equiv \Psi_{ABCD} o^A o^B c^C i^D, \quad *\Psi_2 \equiv \Psi_{ABCD} o^A o^B i^C c^D, \quad \Psi_2^* \equiv \Psi_{ABCD} o^A o^B i^C c^D, \]

\[ \Psi_3 \equiv \Psi_{ABCD} o^A o^B c^C i^D, \quad *\Psi_3 \equiv \Psi_{ABCD} o^A o^B i^C c^D, \quad \Psi_3^* \equiv \Psi_{ABCD} o^A o^B i^C c^D, \]

\[ \Psi_4 \equiv \Psi_{ABCD} i^A i^B i^C i^D, \quad \Psi_4^* \equiv \Psi_{ABCD} i^A i^B i^C i^D, \]

\[ \Psi_{01} \equiv \Psi_{ABCD} o^A o^B \bar{o}^C \bar{i}^D, \quad \Psi_{02} \equiv \Psi_{ABCD} o^A o^B \bar{i}^C \bar{i}^D, \quad \Psi_{12} \equiv \Psi_{ABCD} o^A o^B \bar{i}^C \bar{i}^D. \]

\[ \Psi_{00} \equiv \Psi_{ABCD} o^A o^B \bar{o}^C \bar{o}^D, \quad \Psi_{11} \equiv \Psi_{ABCD} o^A o^B \bar{i}^C \bar{i}^D, \quad \Psi_{22} \equiv \Psi_{ABCD} o^A o^B \bar{i}^C \bar{i}^D. \]

Five 4-D Newman-Penrose components

\[ \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4. \]

Eleven 5-D complex components

\[ *\Psi_0, *\Psi_1, \Psi_1^*, *\Psi_2, \Psi_2^*, *\Psi_3, \Psi_3^*, \Psi_4^*, \Psi_{01}, \Psi_{02}, \Psi_{12}. \]

Three 5-D real components

\[ \Psi_{00}, \Psi_{11}, \Psi_{22}. \]

\[ 2 \times (16 \text{ complex components}) + 3 \text{ real components} = 35. \]
Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- Components of the Ricci spinor:
  \[ \Phi_{00} \equiv \Omega_{ABCD} \tilde{0}^A \tilde{B} \, \tilde{C}^D \, \tilde{D}, \quad \Phi_{11} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{C}^D \, \tilde{D}, \quad \Phi_{22} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{C}^D \, \tilde{D}, \]
  \[ \Omega \equiv \Omega_{ABCD} \tilde{A}^B \tilde{C}^D \, \tilde{D}, \quad *\Phi_{01} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{C}^D \, \tilde{D}, \quad *\Phi_{12} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{C}^D \, \tilde{D}, \]
  \[ \Phi_{01} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{0}^C \tilde{D}, \quad \Phi_{02} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{0}^C \tilde{D}, \quad \Phi_{12} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{0}^C \tilde{D}, \]
  \[ *\Phi_{02} \equiv \Omega_{ABCD} \tilde{A}^B \tilde{0}^C \tilde{D}. \]

- 4-D Newman-Penrose components
  Real: \( \Phi_{00}, \Phi_{11}, \Phi_{22} \), Complex: \( \Phi_{01}, \Phi_{02}, \Phi_{12} \).

- 5-D components
  Real: \( \Omega, \ *\Phi_{01}, \ *\Phi_{12} \), Complex: \( *\Phi_{02} \).

\[ 2 \times (4 \text{ complex components}) + 6 \text{ real components} = 14 \]
Extension of the 4-dimensional Newman-Penrose formalism to 5 dimensions

- **Commutation relations:**

\[
D\Delta - \Delta D = - (\gamma + \bar{\gamma})D - (\epsilon + \bar{\epsilon})\Delta - (\pi + \bar{\pi})\delta - (\bar{\pi} + \tau)\delta + (a + e)D,
\]
\[
D\delta - \delta D = -(\bar{\alpha} + \beta + \bar{\pi})D + \kappa\Delta + (\epsilon - \bar{\epsilon} - \bar{\rho})\delta - \sigma\delta + (\varsigma - \chi)D,
\]
\[
D\bar{D} - \bar{D}D = (-2a + \theta + \bar{\theta})D + 2\delta\Delta + (\bar{\eta} - 2\bar{\chi})\delta + (\eta - 2\chi)\bar{\delta} + fD,
\]
\[
\Delta\delta - \delta\Delta = -\bar{\nu}D + (\bar{\alpha} + \beta + \tau)\Delta + (\gamma - \bar{\gamma} + \mu)\delta + \bar{\lambda}\delta - (\xi + \omega)D,
\]
\[
\Delta\bar{D} - \bar{D}\Delta = -2bD + (2\epsilon - \theta - \bar{\theta})\Delta + (\zeta - 2\bar{\omega})\delta + (\bar{\zeta} - 2\omega)\bar{\delta} - c\bar{D},
\]
\[
\delta\bar{\delta} - \bar{\delta}\delta = (-\mu + \bar{\mu})D + (-\rho + \bar{\rho})\Delta + (-\alpha + \bar{\beta})\delta + (\bar{\alpha} - \beta)\bar{\delta} + (v - \bar{v})D,
\]
\[
\delta\bar{D} - D\delta = (\bar{\zeta} - 2\bar{\xi})D + (\eta + 2\varsigma)\Delta + (\theta - \bar{\theta} - 2\bar{\nu})\delta - 2\phi\bar{\delta} + \psiD.
\]

- **Ricci (Newman-Penrose equations) and Bianchi identities.**
The use of spinors might enable us to simplify tensorial computations in dimension 5.

The extended Newman-Penrose formalism could be used to seek new exact solutions in dimension 5 with prescribed Petrov type.

The extended Newman-Penrose formalism is the starting point for a generalization of the GHP formalism to dimension 5.