Approximate Symmetries and the Energy in Spacetimes ¹

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Abstract: There is a problem of defining energy in time-varying spacetimes as energy is not conserved in them. The problem is particularly severe for gravitational waves as there is no conventional stress-energy tensor. The gravitational field is given by the Weyl tensor. The gravitational field is given by the Weyl tensor but that does not provide a measure of the energy. Pseudo tensors are introduced but they are not invariant quantities. An invariant proxy for a pseudo tensor is defined but it is based on an approximation that would break down in the strong field. Another approach uses the breaking of time translational for a measure of the energy. No attempt has proved unambiguously better than the others invariance in that it not only provides calculational convenience but has definite physical implications. Here we review the use of approximate Lie symmetries of the geodesic equations and the Noether symmetries for the arc length to obtain a definition of energy that does provide physical predictions.

1. Introduction

The energy density of a spacetime is taken to be given by T^{00} . However, this would mean that there is no energy in a vacuum solution of the Einstein equations. Do we have to give up the very useful concept of energy in General Relativity (GR)? To examine this question let us examine the related concepts of momentum and angular momentum in GR.

If a particle is left in a vacuum near a gravitational source it will move. Hence momentum is not conserved there. Again, due to frame-dragging only the azimuthal angular momentum is conserved in the presence of a spinning gravitational source and otherwise angular momentum conservation is lost. Similarly, since gravitational wave spacetimes have no timelike isometry, energy is not conserved in them. Since mass is directly related to energy in Relativity, giving up energy conservation means giving up mass as well. On the other hand, we do need to retain the concept of mass in GR. The status of mass is the most important outstanding problem in GR.

¹Dedicated to the memory of John Archibald Wheeler

An early attempt to deal with this problem was to linearize the Einstein field equations and put the non-linear terms on the right as a source. This was usually done in a gauge-dependent (frame-dependent) way and hence was not generally accepted by relativists. This source was called a *pseudo-tensor* [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. A true tensor can also be defined [11]. There was a claim by Christodoulou [12] that this true tensor missed out the non-linearity of the field but Thorne [13] argued that it had been accounted for in the earlier literature [14, 15]. However, it is based on an approximation that would lead it to break down in a strong gravitational field.

There have been truly invariant attempts at defining energy locally and quasi-locally [16, 17, 18, 19, 20, 21], using twistors, usual tensors and Ashtekar variables. However, all of these are limited in their generality of applicability. Penrose [22] has argued that the energy in the pure gravitational field should be expressible in terms of the invariants of the Weyl tensor. This suggestion has been carried forward analytically and then for use in numerical relativity schemes [23, 24, 25, 26, 27, 28]. However, the energy content of the spacetimes has not actually been computed and so no physical consequences of the proposal are available to be checked.

A different class of approaches has also been adopted, which we will be following up here. Since energy conservation is guaranteed in Classical Mechanics by Noether's theorem if there is time translational invariance [29], one could hope to get a measure of the energy in the spacetime by obtaining a measure of the extent to which time translational symmetry is broken. This suggestion has been followed by various people using different measures [11, 30, 31, 32, 33]. The essence of the measure is to take the magnitude of the deviation from the symmetric and normalize by dividing by the magnitude itself. Though very reasonable, it is still not unique and has not led to physical predictions that can be tested. A wholly different proposal for symmetry breaking has been adopted which will be discussed here. For this purpose the mathematical background will be given briefly in the next section.

Nowhere is the problem of the definition of energy as severely apparent as it is for gravitational wave spacetimes. Following Wheeler's oft repeated advice, to deal with a problem "squarely" rather than approaching it tangentially, one should test the suggestions in this context. The first gravitational wave solution was obtained by Einstein and Rosen [34] for cylindrical waves. The question was raised whether gravitational waves are real in the sense that they can carry energy [35, 36]. To demonstrate that they do it was shown that momentum is imparted to test particles in their path [37, 38, 39]. Much later a general formula for the momentum imparted was obtained [40].

The proposal for defining energy by using approximate Lie and Noether symmetries has been implemented in some well known spacetimes, in some artificially constructed spacetimes and in gravitational wave spacetimes. The method does not allow for delta function plane gravitational waves, which are thus excluded but, it has been applied to cylindrical waves. These developments will be reviewed in the subsequent sections.

2. Isometries, Lie and Noether Symmetries and Approximate Lie and Noether Symmetries

We appeal to Noether or Lie symmetries as Noether's theorem is the basis for associating energy conservation with time translational invariance and the Lie symmetries are the symmetries of the corresponding Euler-Lagrange equations. It states that

Theorem 1. *If $\mathbf{X}$ is a Noether point symmetry for a Lagrangian $L(s, x^i, \dot{x}^i)$ of the Euler-Lagrange equations then,*

$$I = \xi L + (\eta^i - \dot{x}^i \xi) \frac{\partial L}{\partial \dot{x}^i} - A, \quad (1)$$

is a conserved quantity (constant of motion) associated with $\mathbf{X}$ [41, 42]. All Noether symmetries are Lie symmetries but the converse need not be true. For application in Relativity one can use the symmetries of the gravitational (Einstein-Hilbert) Lagrangian, which gives the Einstein field equations, or the metric, which is the Lagrangian giving the geodesic equations. The former does not give the time translational invariance as the energy and hence we consider the latter. The symmetries of the metric tensor, called *isometries* or *Killing vectors* [43, 44], must be Noether and Lie symmetries for this Lagrangian [45].

An *isometry* is a vector field $\mathbf{k}$ along which the metric tensor is invariant under Lie transport [29, 46],

$$\mathcal{L}_{\mathbf{k}} \mathbf{g} = 0. \quad (2)$$

A *Noether symmetry* for a Lagrangian L is defined as [47, 48]

$$\mathbf{X}^{[1]} L + (D_s \xi) L = D_s A, \quad (3)$$

where A is a *gauge function*

$$D_s = \partial / \partial s + \dot{x}^i \partial / \partial x^i \quad (i = 1, \dots, n), \quad (4)$$

the summation convention (that repeated indices are summed over the entire range) has been used and

$$\mathbf{X}^{[1]} = \mathbf{X} + (\eta_{,s}^j + \eta_{,i}^j \dot{x}^i - \xi_{,s} \dot{x}^j - \xi_{,i} \dot{x}^i \dot{x}^j) \partial / \partial \dot{x}^j, \quad (5)$$

is the *prolongation* of the infinitesimal symmetry generator for point transformations

$$\mathbf{X} = \xi(s, x^i) \partial / \partial s + \eta^j(s, x^i) \partial / \partial x^j, \quad (6)$$

the $_{,s}$ referring to the partial derivative relative to s and ξ and η^j being arbitrary functions of the parameter s and the coordinates x^i .

For a second order system of semi-linear ordinary differential equations

$$E^i \equiv \ddot{x}^i - f^i(s, x^j, \dot{x}^j) = 0, \quad (7)$$

$\mathbf{X}^{[1]}$ is a Lie symmetry if

$$\mathbf{X}^{[2]} E^i \big|_{E^i=0} = 0, \quad (8)$$

where

$$\mathbf{X}^{[2]} = \xi \frac{\partial}{\partial s} + \eta^i \frac{\partial}{\partial x^i} + \eta_{,s}^i(s, x^j, \dot{x}^j) \frac{\partial}{\partial \dot{x}^i} + \eta_{,ss}^i(s, x^j, \dot{x}^j, \ddot{x}^j) \frac{\partial}{\partial \ddot{x}^j}. \quad (9)$$

In general a spacetime does not possess exact symmetry but may do so approximately. These approximate symmetries may yield more physical insights than an unbroken symmetry does. Recall that D'Alembert's principle of virtual work provides much greater physical insight than merely taking a static system would do. The exact symmetries form a Lie algebra [29, 46, 47, 48] and the approximate symmetries form an *approximate Lie algebra* [49, 50], in which the algebra does not fully close but is closed to the required approximation. Methods for obtaining exact symmetries and first-order approximate symmetries of a Lagrangian are available [47, 51]. A 1st order approximate Lie symmetry for a system of ODEs, $\mathbf{E}$ is defined by

$$\mathbf{E} = \mathbf{E}_0 + \epsilon \mathbf{E}_1 = O(\epsilon^2), \quad (10)$$

corresponding to a first-order Lagrangian, which is perturbed by ϵ ,

$$L(s, x^i, \dot{x}^i, \epsilon) = L_0(s, x^i, \dot{x}^i) + \epsilon L_1(s, x^i, \dot{x}^i), \quad (11)$$

the functional $\int_V L ds$ is invariant under the one-parameter group of transformations with approximate Lie symmetry generator

$$\mathbf{X} = \mathbf{X}_0 + \epsilon \mathbf{X}_1 + O(\epsilon^3), \quad (12)$$

up to a gauge function

$$A = A_0 + \epsilon A_1, \quad (13)$$

where

$$\mathbf{X}_j = \xi_j \frac{\partial}{\partial s} + \eta_j^i \frac{\partial}{\partial x^i}, \quad (j = 0, 1, i = 0, 1, 2, 3), \quad (14)$$

if

$$\mathbf{X}_0^{[1]} L_0 + (D_s \xi_0) L_0 = D_s A_0, \quad (15)$$

and

$$\mathbf{X}_1^{[1]} L_0 + \mathbf{X}_0^{[1]} L_1 + (D_s \xi_1) L_0 + (D_s \xi_0) L_1 = D_s A_1. \quad (16)$$

$\mathbf{X}_0$ is called the *exact symmetry generator* and $\mathbf{X}_1$ the *first-order approximate part*, L_0 is the *exact Lagrangian* corresponding to the *exact equations* $\mathbf{E}_0 = \mathbf{0}$, and $L_0 + \epsilon L_1$ the first-order approximate Lagrangian corresponding to the first-order perturbed equations $\mathbf{E}_0 + \epsilon \mathbf{E}_1 = \mathbf{0}$. The perturbed equations (13) always have the approximate symmetry generators $\epsilon \mathbf{X}_0$ which are known as *trivial* approximate symmetries and $\mathbf{X}$ given by (12) with $\mathbf{X}_0 \neq \mathbf{0}$ is called a non-trivial approximate symmetry. They yield the exact and first order approximate conserved quantities $I = I_0 + \epsilon I_1$. If I_0 vanishes, then I is called an *unstable approximate first integral* and otherwise it is called *stable*. A detailed discussion on the approximate first integrals for Hamiltonian dynamical system is given in [52].

3. Approximate Symmetries and Energy

One could hope to obtain the energy in the field from the approximate timelike isometry or Noether symmetry, as the loss of energy conservation is presumably due to the energy in the gravitational field. Thus the parameter ϵ , which give the strength of the symmetry breaking should give the amount of energy in the field. Using this definition for the Schwarzschild metric, taking the exact metric to be Minkowski space, the geodesic equations give only trivial approximate symmetries. More specifically, the lost conservation laws are recovered as approximate conservation laws. The amount of non-conservation could be taken to give the energy in the field. There are two non-trivial stable approximate symmetries [53] for the orbital equation.

If one tries to apply the same procedure to the Reissner-Nordström metric it does not work due to the fact that there are *no* approximate symmetries — not even trivial ones. One is forced to use second approximate symmetries [54]. While one *does* then recover the trivial approximate symmetry, one needs to require that the symmetry generator applied to the geodesic is subject to the full (Reissner-Nordström) equations holding and not only the exact (Minkowski) part. For the first approximate symmetries there was no difference as we neglected second order terms. Here there *is* a difference. More interestingly, the time translation generator picks up a scaling factor $(1 - Q^2/2GM^2)$. Thus *the energy is re-scaled by this factor!* This suggests that the energy in the gravitational field may be definable using *second order* approximate symmetries.

The next step was obviously to try to use the above procedure for the Kerr metric. With non-trivial dependence on two variables and the off-diagonal terms in the metric, it becomes very difficult to calculate the symmetry generators. As such, we first worked out the Noether symmetry generators, which are easier to obtain [55]. The energy scaling obtained is more complicated in that it is position dependent, $(1 + a/cr)$. It was not clear how to compare this result with that of the Reissner-Nordstrom metric. Consequently we considered the charged Kerr (or Kerr-Newman) metric. The scaling factor then turns out to be $(1 - Q^2/2GM^2 + a/cr)$. The interesting feature is that the energy depends *linearly* on the spin of the source and only *quadratically* on the charge. As such, the effect of spin is *much stronger*.

We need to deal with spacetimes that do not possess timelike isometries. One could go to an exact gravitational wave solution but it would not then be clear what we should take as the “exact” metric. Instead we can choose a static metric with some symmetries and then make it non-static by introducing a perturbation involving time. It is highly unlikely that the resultant metric would then be a vacuum solution of the Einstein equations. One could determine the stress-energy and the Weyl tensor for the spacetime and interpret it as a wave interacting with matter. Though this procedure suffers from the drawback that the “gravitational wave” and the matter field are artificially constructed, it does provide some insights and guides us on the general significance of the procedure.

We first used this procedure in plane symmetry, taking a space with six isometries [56]

$$ds^2 = e^{2\nu(x)} dt^2 - dx^2 - e^{2\mu(x)}(dy^2 + dz^2), \quad (17)$$

with $\mu(x) = \nu^2(x) = (x/X)^2$, where X is a constant having the same dimensions as x . The perturbed metric was taken as

$$ds^2 = (1 + 2\epsilon t/T)[e^{2\nu(x)} dt^2 - e^{2\mu(x)}(dy^2 + dz^2)] - dx^2. \quad (18)$$

We call this a “gravitational plane wave-like spacetime” [57]. Of course, it is totally non-physical and non-wave-like, in that the perturbation grows linearly with time. However, we *did* get a stable approximate symmetry, which yields the approximate conserved quantity

$$Q = E - \epsilon(tE + yp_y c + zp_z c)/T, \quad (19)$$

as the energy of a particle in the space, if the starting energy-momentum vector is $(E, p_x, p_y c, p_z c)$. However, this does not give the energy of the gravitational field.

We proceeded to obtain the second approximate symmetries and again picked up a scaling factor. The factor is

$$f = t[e^{-4(x/X)} + 2e^{-2(x^2/X^2 + x/X)}]/2T. \quad (20)$$

Hence we get energy definable by re-scaling the energy in the exact spacetime by this factor. Since the example is artificial, this provides useful guidance but one cannot obtain physical insights from it.

We then tried applying the procedure to parallel plane-fronted gravitational waves. The problem with this attempt is that there is no region in the spacetime in which there is curvature, but instead a sudden break. As such, this does not provide much wisdom.

Next we applied the procedure to the cylindrically symmetric static metric with six isometries [58]

$$ds^2 = e^{\nu(\rho)} dt^2 - d\rho^2 - e^{\mu(\rho)}(a^2 d\phi^2 + dz^2), \quad (21)$$

with $\nu(\rho) = (\rho/R)^2$ and $\mu(\rho) = (\rho/R)^3$, where R is a constant having the same dimensions as ρ . We again put in a linear perturbation as before

$$ds^2 = (1 + 2\epsilon t/T)[e^{\nu(\rho)} dt^2 - e^{\mu(\rho)}(a^2 d\phi^2 + dz^2)] - d\rho^2, \quad (22)$$

and obtained a stable approximate symmetry giving an approximate conservation law for the energy-momentum of a test particle. Of course, this does not give the energy in the gravitational field either.

We obtained the second approximate symmetries and the scaling factor in this case as well. It turns out to be

$$f = t[e^{-2(\rho/R)^2} + 2e^{-(\rho/R)^2(\rho/R-1)}]/2T. \quad (23)$$

4. Approximate Symmetries and the Energy in Cylindrical Gravitational Waves

The key requirements of an exact solution of the vacuum Einstein equations and that it have physical significance that could lead to a clear interpretation and possibly a test of the proposal, are met for the cylindrical gravitational waves of Einstein and Rosen [59]

$$ds^2 = e^{2(\gamma-\psi)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi} d\phi^2 - e^{2\psi} dz^2, \quad (24)$$

where γ and ψ are arbitrary functions of t and ρ , subject to the vacuum field equations

$$\psi'' + \frac{1}{\rho}\psi' - \ddot{\psi} = 0, \quad \gamma' = \rho(\psi'^2 + \dot{\psi}^2), \quad \dot{\gamma} = 2\rho\dot{\psi}\psi', \quad (25)$$

dot denotes differentiation with respect t and prime with respect to ρ . The solution of (35) is given by [37]

$$\begin{aligned} \psi &= AJ_0(\omega\rho) \cos(\omega t) + BY_0(\omega\rho) \sin(\omega t), \\ \gamma &= \frac{1}{2}\omega\rho[(A^2 J_0 J_0' - B^2 Y_0 Y_0') \cos(2\omega t) - AB\{(J_0 Y_0' + Y_0 J_0') \sin(2\omega t) - 2(J_0 Y_0' - Y_0 J_0')\omega t\}]. \end{aligned} \quad (26)$$

This metric has two KVs $\partial/\partial\phi$ and $\partial/\partial z$ [43]; this means that there is only azimuthal angular momentum conservation and linear momentum conservation along z .

Here we take the “exact” static spacetime by removing the t -dependent part and putting the strength of the wave $A = 1$. Since Y_0 is badly behaved at $\rho = 0$, we choose $B = 0$ (as the units of the strength are arbitrary),

$$ds^2 = e^{2(\gamma_0-\psi_0)}(dt^2 - d\rho^2) - \rho^2 e^{-2\psi_0} d\phi^2 - e^{2\psi_0} dz^2, \quad (27)$$

where

$$\psi_0 = J_0(\omega\rho), \quad \gamma_0 = \frac{\omega\rho}{2} J_0(\omega\rho) J_0'(\omega\rho) \quad (28)$$

and the strength of the “approximate” wave as a small parameter, i.e. $A = \epsilon$,

$$\begin{aligned} \psi &= J_0(\omega\rho)[1 + \epsilon \cos(\omega t)] = \psi_0 + \epsilon\psi_1 \\ \gamma &= \frac{\omega\rho}{2} J_0(\omega\rho) J_0'(\omega\rho)[1 + \epsilon^2 \cos(2\omega t)] = \gamma_0 + \epsilon^2\gamma_1. \end{aligned} \quad (29)$$

Thus the exact, t -dependent, cylindrical wave becomes the “approximation” to the “exact”, static, spacetime for our purpose.

The scaling factor picked up is [60]

$$f = \dot{\gamma}_1(\dot{t}^2 + \dot{\rho}^2) + 4e^{2(\psi_0-\gamma_0)}\psi_1\dot{\psi}_1\dot{z}^2 - 2\gamma_1'\dot{t}\dot{\rho}, \quad (30)$$

which reduces to

$$\dot{\gamma}_1 e^{2(\psi_0 - \gamma_0)} [e^{2(\psi_0 - \gamma_0)} + e^{3(\psi_0 - \gamma_0)} - 1] - 2\gamma'_1 e^{3(\psi_0 - \gamma_0)} (e^{3(\psi_0 - \gamma_0)} - 1)^{1/2}, \quad (31)$$

for our calculation, assuming that there are no initial velocities in the z and ϕ directions. This calculation includes the Christodoulou “memory effect” [13, 61] as the wave does not contribute any velocity in these directions. This has an asymptotic behaviour [62]

$$\frac{3 \times 2^{11/4}}{\pi^{3/2}} [(|\cos(\omega\rho)|)^{3/2} \sin(2\omega t)] (\omega\rho)^{-1/2} + O([\omega\rho]^{-3/2}). \quad (32)$$

Hence there is a damping of the wave by this factor.

5. Summary and Discussion

We discussed the problem of energy in spacetimes. There have been other attempts to do so. Most have had problems of gauge dependence and some have had problems of approximation that may seriously interfere with their use where points of principle are involved, such as whether gravitational waves are enhanced or attenuated by nonlinear effects. A truly invariant definition using the Weyl tensor *has* been proposed [23, 24, 25, 26, 27, 28] but it does not yield physical predictions. Here the use of approximate Lie symmetries has been advocated on the grounds that the size of the perturbation is irrelevant and so the limit of the perturbation going to zero can be taken. It yields a clear physical prediction that the classical “energy” of the gravitational wave will be attenuated. For exact cylindrical gravitational waves the definition proposed gives the re-scaling factor given in (32).

The main limitation of the procedure used in our definition of the effective “energy” in the spacetime is that it will not apply to a spacetime with a sudden jump from a flat spacetime metric to another flat spacetime metric with a delta function in between. It requires an *interval* over which the metric changes. Of course, it would be necessary to check the robustness of the results on many more metrics than has been done so far. Most importantly, it would be necessary to test the physical predictions made using this definition.

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