Stability analysis of scalar-tensor Born-Infeld black hole solutions

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Phases of scalar-tensor black holes coupled to nonlinear electrodynamics

- We will consider asymptotically flat scalar-tensor black holes coupled to nonlinear electrodynamics. The field equations in Einstein frame are

\[
R_{\mu\nu} = 2\partial_{\mu}\varphi \partial_{\nu}\varphi + 2V(\varphi)g_{\mu\nu} - 2\partial_X L(X,Y) \left( F_{\mu\beta} F_{\nu}^{\beta} - \frac{1}{2} g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right) \\
-2A^4(\varphi)g_{\mu\nu} \left[ L(X,Y) - Y\partial_Y L(X,Y) \right],
\]

\[
\nabla_\mu \left[ \partial_X L(X,Y) F_{\mu\nu} + \partial_Y L(X,Y) (\star F)^{\mu\nu} \right] = 0,
\]

\[
\nabla_\mu \nabla^\mu \varphi = \frac{dV(\varphi)}{d\varphi} - 4\alpha(\varphi)A^4(\varphi) \left[ L(X,Y) - X\partial_X L(X,Y) - Y\partial_Y L(X,Y) \right]
\]

- The black holes have zero electrical charge and nonzero magnetic. The truncated Born-Infeld Lagrangian is

\[
L_{BI}(X) = 2b \left[ 1 - \sqrt{1 + \frac{X}{b}} \right]
\]

- The coupling function defining the considered scalar-tensor theory is

\[
A(\varphi) = e^{\frac{1}{2}\beta\varphi^2}
\]
• We study spherically symmetric static charged black holes so the field equations can be reduced to the following system of ordinary differential equations

\[
\frac{d\delta}{dr} = -r \left( \frac{d\varphi}{dr} \right)^2,
\]

\[
\frac{dm}{dr} = r^2 \left[ \frac{1}{2} f \left( \frac{d\varphi}{dr} \right)^2 - A^4(\varphi)L(X) \right],
\]

\[
\frac{d}{dr} \left( r^2 f \frac{d\varphi}{dr} \right) = r^2 \left\{ -4\alpha(\varphi) A^4(\varphi) [L(X) - X \partial_X L(X)] - rf \left( \frac{d\varphi}{dr} \right)^3 \right\}
\]

• The system admits more that one black hole solution even if we fix the mass and the magnetic charge and specify the boundary conditions [Stefanov, Yazadjiev, Todorov MPLA(2008)]

• The different solutions can be classified by an additional parameter – the scalar charge $D$
Our first step is to study the linear stability of the trivial solutions (solutions with zero scalar field).

We impose linear perturbations of the metric, scalar and electromagnetic fields in the fields equations and the equations for the perturbation of the scalar field is decoupled from the others because of the specific scalar-tensor theory used.

The perturbations of the metric and the electromagnetic field are the same as in the general relativity and are already studied [Moreno, Sarbach PRD(2003), Breton PRD(2005), Fernando, Holbrook IJTP(2006)] so we will consider only the equation for the perturbation of the scalar field. We obtain the following equation

\[ \nabla_\mu^{(0)} \nabla_\mu^{(0)} \varphi = -4\beta \Delta \varphi [L(X) - X \partial_X L(X)] \]

When we separate the variables \( \varphi = \chi(r) e^{i\omega t} Y_{lm}(\theta, \phi) \) and set \( \chi(r) = \frac{\psi(r)}{r} \)

we obtain the following equation

\[ f(r) \frac{d}{dr} \left( f(r) \frac{d\psi(r)}{dr} \right) + \left[ \omega^2 - U(r) \right] \psi(r) = 0 \]

where the potential is \( U(r) = f \left[ \frac{1}{r} \frac{df}{dr} + \frac{l(l+1)}{r^2} - 4\beta [L(X) - X \partial_X L(X)] \right] \)
• The domain of integration is \( r \in [r_H, \infty) \)
• From physical analysis we can determine the boundary condition at infinity

\[
\lim_{r \to \infty} \psi(r) = 0
\]

• The left boundary condition comes from the fact that the perturbation of the scalar field should be finite on the horizon

\[
\psi(r_H) = 0
\]

• We have singular Sturm-Liouville problem because the function \( f(r) \) vanishes on the horizon so the left boundary is singular and the domain of integration is semi-infinite.
Numerical solution of the obtained equations

- In order to determine if a black hole solution is stable or not we have to find the minimal eigenvalue of the resulting Sturm-Liouville problem and more specifically we have to determine if it is positive or negative. If the minimal eigenvalue is positive the solution is stable, if it is negative – unstable.

- The minimal eigenvalue will be greater that the minimum of the potential so it is enough to consider only the potentials that has negative part [Wald JMP(1979,1980)].

- The problem is solved numerically using a modification of the shooting method [Price, “Numerical Solution of Sturm-Liouville Problems”]
Results

Here we show results only for $l=0$ because we obtained that this case have the major contribution to the instability of the black holes.
Preliminary results for the stability of nontrivial solution

- As a first step in studying the linear stability of the nontrivial solutions (solutions with nonzero scalar field) we studied only radial perturbations.
- We expect that as in the trivial case the radial perturbations will have the biggest contribution to the instability.
- The perturbation of the electromagnetic field is zero and the perturbation of the metric can be express using the perturbation of the scalar field. So again is it enough to consider only the perturbation of the scalar field.
- After separation of variables $\Delta \varphi(r,t) = \chi(r)e^{i\omega t}$ and the substitution $\chi(r) = \frac{\psi(r)}{r}$ the following equation is obtained [Stefanov, Yazadjiev, Todorov CQG(2009)]

$$f(r)e^{-\delta(r)} \frac{d}{dr} \left( f(r)e^{-\delta(r)} \frac{d\psi(r)}{dr} \right) + \left[ \omega^2 - U(r) \right] \psi(r) = 0$$

$$U(r) = f(r)e^{-2\delta(r)} \left\{ U(r) + \frac{1 - f(r)}{r^2} + 2A^4(\varphi)L(X) \right\}$$

- This is again a singular Sturm-Liouville problem and it is not much different from the trivial case so the same strategy is used to solve it numerically.
Preliminary results for the stability of nontrivial solution

Outer branches – completely stable:

Middle branches – completely unstable:

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• When we apply the turning point method [Arcioni, Lozano-Tellechea PRD(2005) and references therein] in our case it turns out that the change of stability can only occur at turning points or bifurcations of the diagram $M - T^{-1}$.

• The method can hide a lot of uncertainties but it is very simple and powerful.

• It turns out that the results from the thermodynamical stability analysis [Stefanov, Yazadjiev, Todorov MPLA(2008)] are in a very good agreement with the obtained results from the linear stability analysis.
THANK YOU!