Asymptotic structure of topologically massive gravity in spacelike stretched AdS sector

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Talk outline

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- Canonical realization of the asymptotic symmetry
- Sugawara construction
- Concluding remarks

The talk is based on arXiv:0907.0950 (gr-qc).
Introduction

- Topologically massive gravity with a cosmological constant $\Lambda$, denoted shortly as $\text{TMG}_\Lambda$, is an extension of three-dimensional general relativity with a cosmological constant ($\text{GR}_\Lambda$) by a gravitational Chern-Simons term.
- While $\text{GR}_\Lambda$ is a topological theory, $\text{TMG}_\Lambda$ is a dynamical theory with one propagating mode, the massive graviton.
- In the AdS sector (with $\Lambda < 0$), $\text{TMG}_\Lambda$ contains a maximally symmetric vacuum solution, known as $\text{AdS}_3$, and BTZ black hole, with interesting thermodynamic properties.
- The interpretation of $\text{TMG}_\Lambda$ for generic values of the Chern-Simons coupling constant suffers from serious difficulties:
  - for $G > 0$, massive excitations about $\text{AdS}_3$ carry negative energy;
  - for $G < 0$ the energy of the BTZ black hole is negative.
Li et. al. introduced the so-called chiral version of the theory and argued that it might lead to a consistent theory at both classical and quantum level.

W. Li, W. Song and A. Strominger, JHEP 0804 (2008), 082.

However, TMG has a rather rich vacuum structure. Since the AdS sector of TMG around AdS$_3$ is not consistent, Anninos et al. proposed to choose a new vacuum, the so-called spacelike stretched AdS$_3$, which could be a stable ground state of the theory.


This choice reduces the isometry group $SL(2, R) \times SL(2, R)$ of AdS$_3$ to its four parameter subgroup $U(1) \times SL(2, R)$.

Exploring thermodynamic properties of the spacelike stretched black hole, Anninos et al were led to a hypothesis that the corresponding boundary dynamics is described by a holographically dual two-dimensional conformal field theory.
As a natural step toward verification of the above hypothesis, Compère et al investigated asymptotic symmetries in the spacelike stretched AdS$_3$ sector. They found a structure isomorphic to the semi-direct sum of the $u(1)$ Kac-Moody algebra and the Virasoro algebra, $u(1)_K M \oplus_{sd} V$, with a central extension.


We also examine the correctness of the above hypothesis. Our approach is based on Dirac’s constraint Hamiltonian formalism, in a form applied recently to TMG$_\Lambda$


After formulating a set of natural asymptotic conditions that generalize the usual AdS conditions, we find that the asymptotic symmetry of the spacelike stretched AdS sector of TMG$_\Lambda$ is indeed a two-dimensional conformal symmetry with central charges, in complete agreement with the hypothesis of D. Anninos.
Spacelike stretched black holes

- Instead of using the standard Riemannian formalism, with an action defined in terms of the metric we with find it more convenient to work in the first-order formalism.

- Such an approach can be naturally described in the framework of Poincaré gauge theory, where basic gravitational variables are the *triad field* $b^i$ and the *Lorentz connection* $A^{ij} = - A^{ji} := - \varepsilon^{ij} K \omega^k$ (1-forms), and the corresponding field strengths are the *torsion* $T^i = \nabla b^i$ and the *curvature* $R^i = d \omega^i + \frac{1}{2} \varepsilon_{ijk} \omega^j \omega^k$ (2-forms).

- The Lagrangian of TMG$_{\Lambda}$ is defined by

$$L = 2ab^i R_i - \frac{\Lambda}{3} \varepsilon_{ijk} b^i b^j b^k + a \mu^{-1} L_{CS}(\omega) + \lambda^i T_i ,$$  

where $a = 1/16\pi G$, $L_{CS}(\omega) = \omega^i d\omega_i + \frac{1}{3} \varepsilon_{ijk} \omega^i \omega^j \omega^k$ is the Chern-Simons Lagrangian for the Lorentz connection, and $\lambda^i$ (1-form) is the Lagrange multiplier that ensures $T_i = 0$. 
By construction $\text{TMG}_\Lambda$ is invariant under the local Poincaré transformations: translations parametrized by $\xi^\mu$ and Lorentz rotations parametrized by $\theta^i$.

The gravitational field equations reduce to:

$$ aG_{ij} - \Lambda \eta_{ij} + a\mu^{-1}C_{ij} = 0, $$

where $G_{ij}$ is the Einstein tensor, and $C_{ij} = \varepsilon_i{}^{mn} \nabla_m L_{nj}$ the Cotton tensor.

The spacelike stretched black hole is a particular solution of $\text{TMG}_\Lambda$ with several attractive features:

- it is a discrete quotient of the spacelike stretched $\text{AdS}_3$ vacuum with the same type of asymptotic behaviour;
- the black hole thermodynamics seems to support the hypothesis which "predicts" the existence of an asymptotic conformal symmetry in this sector of $\text{TMG}_\Lambda$. 
The spacelike stretched black hole metric is:

\[
\begin{align*}
ds^2 &= N^2 dt^2 - B^{-2} dr^2 - K^2 (d\varphi + N \varphi dt)^2, \\
4K^2 N^2 &= (\nu^2 + 3)(r - r_+)(r - r_-), \quad \ell^2 B^2 = 4N^2 K^2, \\
4K^2 &= r \left[ 3(\nu^2 - 1)r + (\nu^2 + 3)(r_+ + r_-) - 4\nu \sqrt{r_+ r_- (\nu^2 + 3)} \right], \\
2K^2 N\varphi &= 2\nu r - \sqrt{r_+ r_- (\nu^2 + 3)}, \quad \Lambda = -\frac{a}{\ell^2}, \quad \nu = \frac{\mu \ell}{3}, \quad \nu^2 > 1.
\end{align*}
\]

We choose the triad field in the simple diagonal form:

\[
\begin{align*}
b^0 &= N dt, \quad b^1 = \frac{\ell}{2NK} dr, \quad b^2 = K (d\varphi + N \varphi dt). \quad (3a)
\end{align*}
\]

Consequently, the Levi-Chivita connection $\omega^i$ is:

\[
\begin{align*}
\omega^0 &= -\frac{N \nu}{\ell} dt - \frac{2KK'}{\ell} d\varphi, \quad \omega^1 = -\frac{KN'}{2N} dr, \\
\omega^2 &= -\frac{KN\varphi}{\ell} dt + \frac{K^3 N'}{\ell} d\varphi.
\end{align*}
\]
Asymptotic conditions

- Let us introduce the concept of *spacelike stretched AdS asymptotic behavior*, based on the following requirements:
  - asymptotic configurations should include spacelike stretched black hole geometries;
  - they should be invariant under the action of $U(1) \times SL(2, R)$;
  - asymptotic symmetries should have well-defined canonical generators.
- We adopt the following asymptotic form of the metric:

  $$g_{\mu\nu} = \bar{g}_{\mu\nu} + G_{\mu\nu}, \quad G_{\mu\nu} = \begin{pmatrix} \mathcal{O}_1 & \mathcal{O}_2 & \mathcal{O}_0 \\ \mathcal{O}_2 & \mathcal{O}_3 & \mathcal{O}_1 \\ \mathcal{O}_0 & \mathcal{O}_1 & \mathcal{O}_{-1} \end{pmatrix},$$  \hspace{1cm} (4)

  where $\bar{g}_{\mu\nu}$ is the black hole vacuum.
- The form of the metric can be used to "derive" asymptotic behaviour of the fields in the first-order formalism.
The subset of gauge transformations that leave adopted asymptotic conditions invariant is:

\[ \xi^0 = \ell T(\varphi) + O_2, \quad \xi^1 = -r \partial_2 S(\varphi) + O_0, \quad \xi^2 = S(\varphi) + O_2, \]

\[ \theta^0 = -\frac{2\ell}{\sqrt{3(\nu^2 + 3)(\nu^2 - 1)r}} \partial_2^2 S(\varphi) + O_2, \quad \theta^1 = \frac{2\ell\sqrt{\nu^2 + 3}}{3(\nu^2 - 1)r} \partial_2 T(\varphi) + O_3, \]

\[ \theta^2 = -\frac{4\ell\nu}{(\nu^2 + 3)\sqrt{3(\nu^2 - 1)r}} \partial_2^2 S(\varphi) + O_2. \]

The commutator algebra of the asymptotic transformations takes the form \( u(1)_{KM} \oplus sd V \),

\[ i [k_m, k_n] = 0, \]

\[ i [k_m, \ell_n] = mk_{m+n}, \]

\[ i [\ell_m, \ell_n] = (m - n)\ell_{m+n}. \]
Canonical realization of the asymptotic symmetry

- Canonical generators act on dynamical variables via the Poisson bracket (PB) operation and they must have well-defined functional derivatives.

- When this is not the case, the problem can be usually solved by adding suitable surface terms:

\[
\tilde{G} = G + \Gamma, \quad \Gamma := -\int_0^{2\pi} d\varphi (\ell T \mathcal{E}^1 + S \mathcal{M}^1) \quad (7)
\]

- For \(\xi^0 = 1\) and \(\xi^2 = 1\), the values of the surface terms have the meaning of energy and angular momentum.

- For the spacelike stretched black hole we have:

\[
E = \frac{(\nu^2 + 3)}{24G\ell} \left[ r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (3 + \nu^2)} \right]
\]

\[
M = \frac{\nu(\nu^2 + 3)}{96G\ell} \left[ \left( r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (3 + \nu^2)} \right)^2 - \frac{5\nu^2 + 3}{4\nu^2} (r_+ - r_-) \right]
\]
The PB bracket algebra of the improved canonical generators takes the form:

\[ i\{K_m, K_n\} = -\frac{c_K}{12} m \delta_{m,-n}, \]
\[ i\{K_m, L_n\} = m K_{m+n}, \]
\[ i\{L_m, L_n\} = (m - n) L_{m+n} + \frac{c_V}{12} m^3 \delta_{m,-n}, \]  

(8a)

where

\[ c_K = \frac{(\nu^2 + 3) \ell}{G \nu}, \quad c_V = \frac{(5\nu^2 + 3) \ell}{G \nu (\nu^2 + 3)}. \]  

(8b)

The canonical realization of the asymptotic symmetry is given as \( u(1)_K \otimes_{sd} V \), with central charges \( c_K \) and \( c_V \).

The above asymptotic algebra does not describe the conformal symmetry. However, there is a particular construction due to Sugawara, which reveals how the conformal algebra can be reconstructed.
Sugawara construction

- We introduce the new set of generators

\[ L_n^- := L_n - \bar{L}_n, \quad L_n^+ := -\bar{L}_n - i\alpha K_{-n}, \quad \bar{L}_n := -\frac{6}{c_K} \sum_r K_r K_{n-r}, \]

which obey the following PB relations:

\[ i\{L_m^+, L_n^+\} = (m - n) L_{m+n}^+ + \frac{c^+}{12} m^3 \delta_{m,-n}, \quad i\{L_m^+, L_n^-\} = 0, \]

\[ i\{L_m^-, L_n^-\} = (m - n) L_{m+n}^- + \frac{c^-}{12} m^3 \delta_{m,-n}, \quad (9) \]

where \( c^+ := c_K \alpha^2 \).

- The values of \( L_0^\pm \) for the spacelike stretched black hole are:

\[ L_0^+ = \frac{6 G \nu \ell}{\nu^2 + 3} E^2 = \frac{(\nu^2 + 3) \nu}{96 G \ell} \left[ r_+ + r_- - \frac{1}{\nu} \sqrt{r_+ r_- (3 + \nu^2)} \right]^2, \]

\[ L_0^- = L_0^+ - M = \frac{(\nu^2 + 3) (5 \nu^2 + 3)}{384 \nu G \ell} (r_+ - r_-)^2. \quad (10) \]
To find out the value of $\alpha$, one can use our central charges $c^\pm$ to calculate the black hole entropy via Cardy’s formula:

$$S_c = 2\pi \alpha \frac{(\nu^2 + 3)}{24G} \left[ r_+ + r_- - \frac{1}{\nu} \sqrt{r+r_- (3+\nu^2)} \right] + \frac{\pi (5\nu^2 + 3)}{24\nu G} (r_+-r_-).$$

The gravitational black hole entropy of $\text{TMG}_\Lambda$ has the form:

$$S_{\text{gr}} = \frac{\pi}{24\nu G} \left[ (9\nu^2 + 3)r_+ - (\nu^2 + 3)r_- - 4\nu \sqrt{r+r_- (\nu^2 + 3)} \right].$$

One finds that $S_c = S_{\text{gr}}$ for $\alpha = \frac{2\nu}{\nu^2 + 3} \equiv \frac{2\ell}{Gc_K}$.

Consequently, the values of the central charges in the Virasoro algebras are the same as those conjectured by D. Anninos et al:

$$c^- = \frac{(5\nu^2 + 3)\ell}{G\nu(\nu^2 + 3)}, \quad c^+ = \frac{4\nu \ell}{G(\nu^2 + 3)}.$$

(11)
Concluding remarks

We analyzed asymptotic structure of TMG$_\Lambda$ in the spacelike stretched AdS sector:

- We introduced spacelike stretched AdS asymptotic conditions and found that the commutator algebra of the asymptotic transformations has the form $u(1)_ {KM} \oplus_{sd} V$.
- With the adopted asymptotic conditions, we constructed the improved canonical generators and found the expressions for the conserved charges. We showed that canonical algebra of the improved generators takes the form $u(1)_ {KM} \oplus_{sd} V$ with two central charges.
- We used the Sugawara construction in the $u(1)_ {KM}$ sector to show that the asymptotic dynamics of TMG$_\Lambda$ can be described by the conformal symmetry, realized by two independent Virasoro algebras with different central charges. This result proves that the hypothesis formulated by D. Anninos et al is correct, at least at the classical level.