Analytic Study of Odd Parity Perturbations of the Self Similar LTB Spacetime

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Supported by the Irish Research Council for Science, Engineering and Technology
Outline

1. Introduction
2. The LTB Spacetime
3. GS Formalism
4. Behaviour on Cauchy Horizon
5. Even Parity Perturbations
6. Conclusion
The background metric and stress energy are

\[ d\bar{s}^2 = -dt^2 + e^\nu(t, r) \, dr^2 + R^2(t, r) \, d\Omega^2 \]  
\[ \bar{T}^{\mu\nu} = \bar{\rho}(t, r) \, \bar{u}^{\mu} \, \bar{u}^{\nu} \]  

We now assume that the LTB spacetime is self-similar, that is, that it possesses a homothetic Killing vector field \( \xi \) such that

\[ \mathcal{L}_\xi g_{\mu\nu} = 2g_{\mu\nu} \]  

This results in \( m(r) = \lambda r \) where \( m(r) \) is the Misner-Sharp mass.

The parameter \( \lambda \) is constrained to \( 0 < \lambda \leq 0.09 \) for nakedness.
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Gerlach-Sengupta Formalism

(Gerlach & Sengupta (1979)) We split the spacetime into two submanifolds $M^2(x^A)$ and $S^2(x^a)$.

The odd parity metric perturbations are

$$\delta g_{AB} = 0 \quad \delta g_{Ab} = h_A^0 S_b$$

$$\delta g_{ab} = h (S_{a:b} + S_{b:a})$$

The odd parity matter perturbations are

$$\delta t_{AB} = 0 \quad \delta t_{Ab} = \Delta t_A^0 S_b \quad \delta t_{ab} = 2\Delta t S_{(a:b)}$$

The gauge invariants are

$$k_A = h_A - h\rvert_A + 2h\frac{R\rvert_A}{R}$$

$$L_A = \Delta t_A - Q h_A \quad L = \Delta t - Q h$$
The Master Equation

\[ \beta(z) \frac{\partial^2 A}{\partial z^2} + \gamma(z) \frac{\partial^2 A}{\partial p^2} + \xi(z) \frac{\partial^2 A}{\partial z \partial p} \]

\[ + a(z) \frac{\partial A}{\partial z} + b(z) \frac{\partial A}{\partial p} + c(z) A = e^{\kappa p \Sigma} \quad (9) \]

where

\[ a(z) = 2ze^{-\nu}(2 - \kappa) + \frac{\dot{\nu}}{2}(1 + z^2e^{-\nu}) - \frac{2\dot{S}}{S} \beta(z), \]

\[ b(z) = e^{-\nu}(2\kappa - 5) - e^{-\nu}z \left( \frac{\dot{\nu}}{2} + \frac{2\dot{S}}{S} \right), \]

\[ c(z) = -e^{-\nu}(\kappa^2 - 5\kappa + 4) + ze^{-\nu} \left( \frac{\dot{\nu}}{2} + \frac{2\dot{S}}{S} \right) \left( \kappa - 4 \right) + \mathcal{L}S^{-2}, \]

\[ \beta(z) = 1 - z^2e^{-\nu}, \quad \xi(z) = 2ze^{-\nu} \]

\[ \Sigma = -16\pi e^{-\nu/2}S^2 \partial_r(\bar{\rho}U), \quad \gamma(z) = -e^{-\nu} \]
Penrose Diagram for Self Similar LTB Spacetime

- $z = z_i$
- $t = t^*$
- $z_p$
- $R = 0$
- $z_c$
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Theorem 1

- Symmetric Hyperbolic form of (9):

\[ \vec{\Phi}, \bar{z} = X \vec{\Phi}, p + Y \vec{\Phi} + \vec{j} \]  \hspace{1cm} (10)

where \( \bar{z} := \int_{z_i}^{z} ds \beta(s)^{-1} \).

- The initial data surface is \( S_i = \{(z_i, p)| \ z_i = 0, p \in \mathbb{R}\} \).

Theorem 1: Existence and Uniqueness

Let \( \vec{f}, \vec{g} \) and \( \vec{j} \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R}^3) \). Then there exists a unique solution \( \vec{\Phi}(z, p), \vec{\Phi} \in \mathcal{C}^\infty(\mathbb{R} \times (z_c, z_i], \mathbb{R}^3) \), to the initial value problem consisting of (10) with the initial conditions

\[ \vec{\Phi}|_{z_i} = \vec{f} \quad \vec{\Phi}, z |_{z_i} = \vec{g} \]  \hspace{1cm} (11)

For all \( z \in (z_c, z_i] \) the vector function \( \vec{\Phi}(z, \cdot) : \mathbb{R} \rightarrow \mathbb{R}^3 \) has compact support.
First Energy Norm

- Define the energy norm

\[ E_1(\bar{z}) = E_1[A](\bar{z}) := \int_{\mathbb{R}} \| \Phi \|^2 dp \]  \hspace{1cm} (12)

**Corollary 1: Bound on \( E_1(\bar{z}) \)**

\( E_1[A](\bar{z}) \) is differentiable on \([0, \infty)\) and satisfies the bound

\[ E_1[A](\bar{z}) \leq e^{B_0 \bar{z}} \left( E_1[A](0) + \int_0^\infty \| j \|_2^2 dp \right) \]  \hspace{1cm} (13)

where \( B_0 = \sup_{\bar{z} > 0} | l - 2Y | < \infty \).
Second Energy Norm

- We define another energy integral

\[
E_2[A](z) := \int_{\mathbb{R}} \beta(z) A_z^2 \gamma(z) A_p^2 + H(z) A^2 + K(z) e^{2\kappa p} \Sigma^2 \, dp
\]

which is positive definite by construction.

**Theorem 2: Bound on \(E_2(z)\)**

Let \(A(z, p)\) be a solution to (9) which is subject to Theorem 1. Then for a choice of constant \(\kappa \in [0, \kappa_{min}^*]\), where \(\kappa_{min}^* = 9/4\), constant \(\mu > 0\) and a choice of function \(K(z)\), the energy \(E_2(z)\) of \(A(z, p)\) obeys the *a priori* bound

\[
E_2(z) \leq C_1 E_1[A](0) + C_2 J_{\kappa}[\Sigma_i] \quad (15)
\]

where \(J_{\kappa}[\Sigma_i] = \int_{\mathbb{R}} e^{2\kappa p} \Sigma_i \, dp\), \(z \in (z_c, z_i]\) and \(C_1\) and \(C_2\) are constants.
Theorems 3 and 4

Theorem 3: Bound on $A(z, p)$

Let $A(z, p)$ be a solution to (9) which is subject to Theorem 1 and choose the same constants and functions as in Theorem 2. Then $A(z, p)$ is uniformly bounded on $(z_c, z_i], \ p \in \mathbb{R}$. That is, there exists constants $C_1 > 0, \ C_2 > 0$ such that

$$|A(z, p)| \leq C_1 E_1(0) + C_2 J_\kappa[\Sigma_i] \quad (16)$$

Theorem 4: Behaviour of $A(z, p)$ on Cauchy Horizon

Let $A(z, p)$ be a solution of (9) subject to Theorem 1 and choose the same constants and functions as in Theorem 2. Then $A_{\mathcal{H}+} := \lim_{z \to z_c} A(z, \cdot) \in \mathcal{C}^\infty(\mathbb{R})$ obeys the bound

$$|A_{\mathcal{H}+}(z, p)| \leq C_1 E_1[A](0) + C_2 J_\kappa[\Sigma_i] \quad (17)$$
Theorem 5: General Initial Data

Let $\kappa \in [0, \kappa_{\min})$. Let $f \in H^{1,2}(\mathbb{R})$, $g \in L^2(\mathbb{R})$, $\Sigma \in L^2(\mathbb{R})$. Then there exists a unique solution $A \in C((z_c, z_i] \times \mathbb{R})$ of the initial value problem consisting of (9) with the initial data $A|_{z_i} = f$, $A_z|_{z_i} = g$. This solution satisfies the a priori bound

$$|A(z, p)| \leq C_0 E_1[A](0) + C_1 J_{\kappa}[\Sigma_i]$$

(18)

for $z \in (z_c, z_i]$ and $p \in \mathbb{R}$. 
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The even parity metric perturbations are

\[ \delta g_{AB} = h_{AB} Y \quad \delta g_{Ab} = h^E_A Y : b \]  \hspace{1cm} (19)

\[ \delta g_{ab} = R^2 K Y \gamma_{ab} + R^2 G Z_{ab} \]  \hspace{1cm} (20)

The even parity matter perturbations are

\[ \delta t_{AB} = \Delta t_{AB} Y \quad \delta t_{Ab} = \Delta t^E_A Y : b \]  \hspace{1cm} (21)

\[ \delta t_{ab} = R^2 \Delta t^3 \gamma_{ab} Y + R^2 \Delta t^2 Z_{ab} \]  \hspace{1cm} (22)

The even parity gauge invariant metric perturbations are

\[ k_{AB} = h_{AB} - (p_{A|B} + p_{B|A}) \quad k = K - 2 \frac{R^{|A}}{R} p_A \]  \hspace{1cm} (23)
The even parity gauge invariant matter perturbations are

\[ T_{AB} = \Delta t_{AB} - t_{AB}^C p_C - 2 (t_{CA} p_C^{|B} + t_{CB} p_C^{|A}) \] (24)

\[ T_A = \Delta t_A - t_A^C p_C - R^2 \left( \frac{t_a^a}{4} \right) G_{|A} \] (25)

\[ T^3 = \Delta t^3 - \frac{p_C}{R^2} \left( \frac{R^2 t_a^a}{2} \right)_{|C} \quad T^2 = \Delta t^2 - \left( \frac{R^2 t_a^a}{2} \right) G \] (26)

The first order reduction variables are
\[ X = (\alpha(z, p), \beta(z, p), k(z, p), \dot{k}(z, p), k'(z, p), \Gamma(z, p)), \] which obey
\[ A(z) \dot{X} + B(z) X' + C(z) X = \Sigma \] (27)
First Order Reduction

- The non-trivial constraint takes the form
  \[ F(z) X' + G(z) X = S \]  \hspace{1cm} (28)

- The free evolution system is given by
  \[ \dot{\hat{K}} + H(z) \hat{K}' + C(z) \hat{K} = \tilde{\Sigma} \]  \hspace{1cm} (29)

which can be rewritten as
  \[ t \frac{\partial \bar{K}}{\partial t} + t C(z) \bar{K} = t \tilde{\Sigma} \]  \hspace{1cm} (30)

where \( \bar{K} = \int \hat{K}(z, p) \, dp \) and \( t = z - z_c \). This has a fundamental matrix given by

\[ \bar{\chi}(t, p) = \bar{\chi}_0 + \bar{\chi}_p = P(t) t \bar{E}_0 - P(t) t \bar{E}_0 \int_0^t s^{-\bar{E}_0} P^{-1}(s) s \bar{\Sigma} \, ds \]  \hspace{1cm} (31)
We define $S = C_0^\infty(\mathbb{R}, \mathbb{R}^4)$ as our choice of initial data for Eq (29). Define $S'$ to be the set of initial data corresponding to solutions which are finite in the approach to the Cauchy horizon and $S'' = S \setminus S'$

**Theorem 6: $L^q$-norm Blow-Up**

Suppose $X_0 \in S''$. Then the unique solution $X(z, p)$ of Eq (29) with initial data $X_0$ satisfies the blow-up condition

$$\lim_{z \to z_c} \|X(z, p)\|_q = \infty, \quad 1 \leq q \leq \infty$$

(32)

where $\| \cdot \|_q$ denotes the $L^q$-norm.
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Conclusion

- We have shown that an odd-parity perturbation which has finite energy on an initial hypersurface will be finite on the Cauchy horizon.

- The even-parity perturbations display generic $L^q$-norm blow-up behaviour.

- Extensions include a proof of point-wise blow-up behaviour in the even-parity case and an improved choice of initial data surface.
Conclusion

**Figure:** Structure of the Naked Self-Similar LTB spacetime