Gravitational origin of the Pioneer Anomaly in metric theories of gravitation: can it be done without destruction of coherence between the theory and experiment in gravitational physics?

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Introduction
The Pioneer Anomaly
Our objective

Space-time determination from radial and circular motions
Motion in arbitrary coordinates
Coordinate choice and space-time determination

The Pioneer Anomaly, its source in GR and light deflection
The Pioneer Anomaly
Metric perturbation
Metric origin in GR
Light deflection

Conclusions and future prospects
The Pioneer anomaly

- Pioneer 10/11, Galileo, Uliss in outer Solar system.
- Surprisingly «unmodelled» acceleration was found.
- The acceleration is the same for all spaceships and independent on radius.
- Its value is defined most accurately for Pioneer 10
  \[ a_P = 8.74 \pm 1.33 \times 10^{-10} \text{ m/s}^2. \]
- One of the very few experiments in the celestial mechanics non-consistent with General Relativity
- The same value of accelerations suggests metric origin of them.
- At the same time there are no signatures of such an acceleration in the orbits of outer planets and asteroids (L. Iorio et. al., 2007, 2008; K. Tangen, 2007).
Goal

Determination of the space-time, radial motion in which shows Pioneer anomaly without affecting circular orbits.

This possibility follows from the existence of two metric functions in the spherically-symmetric static space-time. Usually only one of them is used, because only change in time component of metric corresponds to Newtonian limit for slowly moving bodies, e. g. planets.
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Spherically-symmetric static space-time — Radial motion

Interval in the general coordinates

\[ ds^2 = e^{\tau(r)} dt^2 - e^{\rho(r)} dr^2 - e^{\sigma(r)} r^2 (d\theta^2 + \cos^2 \theta d\phi^2). \]  

(1)

Radial motion can be obtained from the energy

\[ g_{tt} \frac{dt}{ds} = e^{\tau(r)} u^0 = k = \text{const} \]  

and 4-velocity length conservation

\[ e^{\tau(r)} u^0 u^0 - e^{\rho(r)} u^1 u^1 = \varepsilon, \quad \varepsilon = 0 \text{ or } 1: \]

\[ \frac{dt}{dr} = \frac{e^{\frac{\rho(r) - \tau(r)}{2}}}{\sqrt{1 - \varepsilon e^{\tau(r)} / k^2}}. \]  

(2)
The universal formula for Doppler shift in the geometric optics approximation

\[
\frac{\nu_r}{\nu_e} = \frac{s_e}{s_r} = \frac{\vec{u}_r \cdot \vec{k}_r}{\vec{u}_e \cdot \vec{k}_e},
\]

(3)

\(\nu_r\) and \(\nu_e\) — received and emitted frequencies, measured by the standard observers with atomic clocks, 
\(s_r\) and \(s_e\) — proper time of one circle of oscillation, 
\(u_r\) and \(u_e\) — 4-velocity of receiver and emitter, 
\(k_r\) and \(k_e\) — tangential null vector (wave vector), paralelly transported along the path of signal.
Radial motion in Doppler tracking

The signal is emitted from the «fixed» Earth from \( r = r_0 \), and received on the spaceship, then retranslated to Earth. The finally received on Earth frequency \( \nu_r \) is connected to the initially emitted \( \nu_e \) as

\[
\nu_r = \nu_e \frac{1 - \sqrt{1 - e^{\tau(r)}/k^2}}{1 + \sqrt{1 - e^{\tau(r)}/k^2}} \quad \text{no } e^{\rho(r)} \text{ dependence!} \quad (4)
\]

Red shift: \( z(t) = \frac{\Delta \nu}{\nu} = \frac{2}{1 + 1/\sqrt{1 - e^{\tau(r(t))/k^2}}, \quad (5)}
\]

\[
e^{\tau(r(t))} = k^2 \left[ 1 - \left( \frac{z(t)}{2 - z(t)} \right)^2 \right]. \quad (6)
\]

We can determine \( e^{\tau(r(t))} \), but we don’t know \( r(t) \).
There is a gauge alternative: prescribe \( r(t) \) or \( e^{\rho(r)} \).
Circular motion and radial motion, light propagation time

For the circular motion $\theta = 0$, $\phi = \omega t$ and

$$\omega^2(r) = \frac{(e^{\tau(r)})'}{(r^2 e^{\sigma(r)})'}.$$  \hfill (7)

Radial motion

$$\frac{dt}{dr} = e^{\frac{\rho(r)-\tau(r)}{2}} \frac{e^{\frac{\rho(r)-\tau(r)}{2}}}{\sqrt{1 - \varepsilon e^{\tau(r)}/k^2}}.$$  \hfill (8)

One-way light propagation time

$$t_p = \int_{r_0}^{r} e^{\frac{\rho(r)-\tau(r)}{2}} dr.$$  \hfill (9)

The maximal simplification suggests null coordinates $\tau(r) \equiv \rho(r)$. 
Circular motion and radial motion, light propagation time

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Space-time determination in null coordinates

\[ t_p = r - r_0 \quad \Rightarrow \quad t = \frac{t_r + t_e}{2}, \quad r = r_0 + \frac{t_r - t_e}{2}, \quad (10) \]

\[ \frac{dt}{dr} = \frac{1}{\sqrt{1 - \varepsilon e^{\tau(r)}/k^2}} \quad \Rightarrow \quad k^2 = \frac{1}{1 - v^2} \quad (11) \]

\[ e^{\tau(r(t))} = k^2 \left[ 1 - \left( \frac{z(t)}{2 - z(t)} \right)^2 \right], \quad (12) \]

\[ \omega^2 = \frac{(e^{\tau(r)})'}{(r^2 e^{\sigma(r)})'} \quad \Rightarrow \]

\[ r^2 e^{\sigma(r)} = r_0^2 e^{\sigma(r_0)} - \int_{r_0}^{r} \frac{4k^2 z(r)z'(r)}{(2 - z(r))^3 \omega^2(r)} \, dr. \quad (13) \]
The Pioneer Anomaly

\[
\frac{d}{d^{ET}}(\nu_r - \nu_m) = -\nu_e \frac{2a_P}{c}, \quad (14)
\]

\(\nu_m\) — "modelled" frequency taking into account all known sources of frequency shifts

\(a_P\) — anomalous acceleration

Basic definitions:

Circular motion is the same as in the Schwarzschild field

Radial motion differs slightly from the Schwarzschild field
Schwarzschild space-time: null coordinates and "modelled" values

\[ ds^2 = \frac{W(e^{\frac{r}{r_g}} - 1)}{1 + W(e^{\frac{r}{r_g}} - 1)}(dt^2 - dr^2) - \]

\[ -r_g^2 \left(1 + W(e^{\frac{r}{r_g}} - 1)\right)^2 (d\theta^2 + \cos^2 \theta d\phi^2), \quad (15) \]

\[ v_m = v_0 \frac{1 + 1/W(e^{\frac{r}{r_g}} - 1)}{1 - v^2} \left(1 - \sqrt{1 - \frac{1 - v^2}{1 + 1/W(e^{\frac{r}{r_g}} - 1)}}\right)^2, \quad (16) \]

\[ \left(\frac{dr}{dt}\right)_m = \sqrt{\frac{v^2 + 1/W(e^{\frac{r}{r_g}} - 1)}{1 + 1/W(e^{\frac{r}{r_g}} - 1)}}. \quad (17) \]
Determination of $\delta e^{\tau}(r)$

\[
\delta e^{\tau}(r) \simeq -\frac{z(r) \delta z(r)}{2} = a_P z(r) \Delta t(r), \tag{18}
\]

\[
t_m(r) = t(r_0) + \int_{r_0}^{r} \frac{dr}{r} \simeq \approx t_0 + \frac{r}{\sqrt{v^2 + \frac{r_g}{r}}} - \frac{r_g}{v^3} \sinh^{-1} \left(\sqrt{\frac{r}{r_g}} v\right), \tag{19}
\]

with the relative uncertainty less than $10^{-8}$. For $z(r)$ with the accuracy higher than $6 \cdot 10^{-5}$

\[
z_m(r) = 2r_m = 2\sqrt{v^2 + \frac{r_g}{r}}. \tag{20}
\]
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Time metric function perturbation

\[ \delta e^{\tau(r)} = 2a_P \left( r + \right. \]
\[ + \frac{r_g}{v^2} \left[ 1 - \sqrt{1 + \frac{r_g}{r} \frac{1}{v^2} \sinh^{-1} \left( \sqrt{\frac{r}{r_g}} v \right)} \right] - \]
\[ \left. \left( \frac{r_g}{r} \right) \right) \), \quad (21) \]

Non-linear perturbation, but for the precision of measurements the difference can be neglected and linear approximation of \( \delta e^{\tau(r)} \sim 2\eta a_P (r - r_0) \) will be sufficient
Graph of time metric perturbation

\[ \Delta e^{\tau(r)} \times 10^{14} \]

\[ \delta e^{\tau(r)} \] of the Pioneer anomaly for \( v \) from 5 km/s to 50 km/s in 5 km/s steps (from down to up) for the metric matching to Schwarzschild’s on 12 a. u.
Transversal metric perturbation

In the first approximation

$$\delta e^{\sigma(r)} = \frac{4a_P \eta r_g}{r^2} \int_{r_0}^{r} \left(1 + W\left(e^{r_g/r} - 1\right)\right)^3 \, dr =$$

$$= \frac{4a_P \eta r_g}{r^2} \int_{r_0}^{r} \left(\frac{r}{r_g}\right)^3 \, dr = \frac{4a_P \eta (r^4 - r_0^4)}{r^2 r_g}.$$

Note the gravitational radius of the source $r_g$ in the answer. So the effect of the Pioneer Anomaly can be reproduced only by perturbations of Schwarzschild space-time and not Minkowski one, but can be obtained without the equivalence principle violation required by various authors (L. Iorio et. al., 2007, 2008; K. Tangen, 2007).
Einstein tensor

\[ ds^2 = e^\tau dt^2 - e^\rho dr^2 - e^\sigma (d\theta^2 + \sin^2 \theta d\phi^2) \quad \Rightarrow \]

\[ E_{ij} = \frac{e^{-\rho}}{4} \left( \lambda_t T_i \otimes T_j - \lambda_s S_i \otimes S_j - \lambda g_{ij} \right), \quad (23) \]

\[ S_i = \{0, e^{\frac{\rho}{2}}, 0, 0\}, \quad T_i = \{e^{\frac{\tau}{2}}, 0, 0, 0\}, \quad -S_i S^i = T_i T^i = 1, \]

\[ \lambda_t = 4e^{\rho-\sigma} + (\rho' - 2\sigma' - \tau') (\sigma' - \tau') + 2 (\tau'' - \sigma''), \]

\[ \lambda_s = 4e^{\rho-\sigma} - \tau' (\sigma' - \tau') - \rho' (\sigma' + \tau') + 2 (\tau'' + \sigma'') , \]

\[ \lambda = \sigma'^2 + \sigma' \tau' + \tau'^2 - \rho' (\sigma' + \tau') + 2 (\tau'' + \sigma''). \]

In the limit of \( a/r_g = \text{const} \), \( r_g \ll r \) leading terms give

\[ \lambda_t = -96 \frac{a_P \eta}{r_g}, \quad \lambda_s = -32 \frac{a_P \eta}{r_g} \frac{r_0^4}{r^4}, \]

\[ \lambda = 16 \frac{a_P \eta}{r_g} \left( 3 - \frac{r_0^4}{r^4} \right). \quad (24) \]
Energy-momentum tensor

For \( r \rightarrow \infty \) energy-momentum tensor

\[ T_{ab} = \kappa^{-1} E_{ab} \tag{25} \]

becomes that of ideal fluid

\[ T_{ab} = (\rho + p) u_a u_b - p g_{ab} \tag{26} \]

with constant pressure and density

\[
\rho = \frac{e^{-\tau}}{4\kappa} \lambda \rightarrow 12 \frac{a \rho \eta}{r_g \kappa} > 0,
\]

\[
\rho = \frac{e^{-\tau}}{4\kappa} (\lambda_t - \lambda) \rightarrow -36 \frac{a \rho \eta}{r_g \kappa} < 0.
\]

The EOS is equal to the dark energy with \( w = -1/3 \).
Light deflection — refractive index

can be obtained easily in isotropic coordinates. If

\[
ds^2 = e^{T(r_i)} dt^2 - e^{R(r_i)} \left( dr_i^2 + r_i^2 (d\theta^2 + \cos^2 \theta d\varphi^2) \right),
\]

then refractive index is

\[
n(r_i) = \frac{dt}{dl} = e^{\frac{R(r_i) - T(r_i)}{2}}.
\]

or in arbitrary coordinates

\[
n(r) = \exp \left( \frac{\sigma(r) - \tau(r)}{2} - \int_r^\infty \left( e^{\frac{\rho(r) - \sigma(r)}{2}} - 1 \right) \frac{dr}{r} \right).
\]
Light deflection — declination angle

0-th approximation: \( r \cos \phi = b \).

Total declination angle is

\[
\Delta \alpha = - \int_{\phi_0}^{\phi} (\ln n)'_r \frac{b}{\cos \phi} d\phi = \int_{\phi_0}^{\phi} \left( \frac{r(\tau'(r) - \sigma'(r))}{2} + 1 - e^{\frac{\rho(r) - \sigma(r)}{2}} \right) d\phi = \int_{\phi_0}^{\phi} \frac{b(T'(r_i) - R'(r_i))}{2 \cos \phi} d\phi. \quad (30)
\]

\[
\delta \Delta \alpha = \int_{\phi_0}^{\phi} \left( \frac{r(\delta \tau'(r) - \delta \sigma'(r))}{2} - \frac{\delta \rho(r) - \delta \sigma(r)}{2} \right) d\phi. \quad (31)
\]
Light deflection — perturbation

\[ \delta \Delta \alpha = \frac{2 \eta a_P}{r_g} b l = 0 \div 6'' \]  

(32)

\( l \) is the distance traveled by the light ray in the perturbed field — at least 70 a. u.,  
\( b \) is the impact parameter  
\( b_{max} = 1 \) a.u.

This is too much — light deflection is confirmed to coincide with GR predictions at level of parts of milliarcsecond (Will, 2005).
Conclusion

▶ The perturbation of time metric coefficient is nearly linear in $r$ and for the transversal coefficient is proportional to $r^2$. Non-linearity in $e^{\tau(r)}$ cannot be recovered from the current measurements.

▶ The model proposed must be carefully studied by the «Grand-fit» investigation (Pitjeva E. V., 2005; Standish E. M., 2008), but direct measurements from the planned missions for testing General Relativity in space are preferable.

▶ EOS of matter forming this metric in GR is that of dark energy with $w = -1/3$ and constant $\rho, p$ at space infinity. The exact solution is known, but in a very complicated form (K.P. Stanjukovich, V.N. Mel’nikov, 1983).

▶ Light deflection in the metric is in direct contradiction with observed values. This fact strongly suggest non-gravitational origin of the Pioneer Anomaly.
Thank You for Your kindly attention.